DGS
GPBC

Calculus with Applications
Good Problems

Justin M. Ryan

Mathematics Department
Butler Community College
Andover, Kansas
USA

jryan10@butlercc.edu

DRAFT
13 March 2017

These notes consist of a collection of in-class labs created for my Calc with Apps classes. The problems are "good" for different reasons, depending on the topic of interest.

http://geometerjustin.com/teaching/bc/gp/

Preface

Write Preface here. Philosophy of the problems, book used in class, *etc.*

Instructions

Complete all exercises. You may work in groups or independently, depending on your personal learning taste. Attempt each problem without using a calculator, and only refer to your calculator (physical or online) if instructed by the problem, or if absolutely necessary. If you get stuck on a problem, discuss it with your group and/or neighboring groups. If the group is still stuck, then your entire group may ask me for a hint.

Students are expected to be present and working on these Good Problems in class, during the time allotted by the instructor. Any problems that are not completed during class time are expected to be completed outside of class, before the next class meeting.

This paper will not be collected, but you will need to submit your answers to selected problems on Canvas.

Contents

1. Limits

These Good Problems cover material from section 2.2 of our book. These good problems consist of a mix of computational and conceptual exercises. The goal is to cement the ideas of this week's lecture, with an eye looking forward to next week's lecture.

1. Consider the function $f(x) = \dfrac{x^2 + 3x + 3}{x - 2}$. Compute the following limits, if they exist.

a.) $\lim\limits_{x \to 1} f(x)$

b.) $\lim\limits_{x \to 2} f(x)$

2. Consider the function $f(x) = \dfrac{2x^2 - x - 1}{x - 1}$.

a.) What is the domain of f?

b.) Calculate $\lim\limits_{x \to 1} f(x)$.

3. Consider the function $f(x) = \dfrac{x+7}{x^2+9x+14}$.

a.) Compute $\lim\limits_{x\to -7} f(x)$

b.) Compute $\lim\limits_{x\to -2} f(x)$

Use a graphing utility to graph the function $y = f(x)$. Explain in terms of the graph of $y = f(x)$ why the limits $\lim\limits_{x\to -7} f(x)$ and $\lim\limits_{x\to -2} f(x)$ are fundamentally different.

4. Consider the function $f(x) = x^2$.

a.) What is $f(2)$?

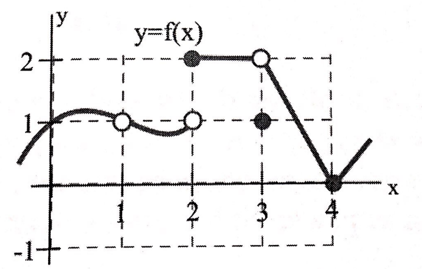
b.) Compute $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$.

5. Consider the function $f(x) = x^6$.

a.) Compute the difference quotient of f, $DQ = \dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$. [Hint: Use Pascal's Triangle (Binomial Theorem) to compute $f(x+h)$.]

b.) Compute $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

6. Consider the following graph.



Compute the following,

i.) $f(1) =$

ii.) $\lim_{x \to 1} f(x) =$

iii.) $f(3) =$

iv.) $\lim_{x \to 3} f(x) =$

v.) $f(2) =$

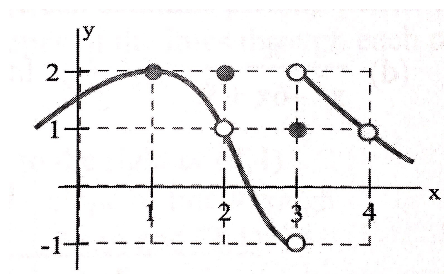
vi.) $\lim_{x \to 2^-} f(x) =$

vii.) $\lim_{x \to 2^+} f(x) =$

viii.) $\lim_{x \to 2} f(x) =$

ix.) At what x-values is the function discontinuous?

7. Consider the following graph.



Compute the following,

i.) $f(2) =$

ii.) $\lim_{x \to 2} f(x) =$

iii.) $f(4) =$

iv.) $\lim_{x \to 4} f(x) =$

v.) $f(3) =$

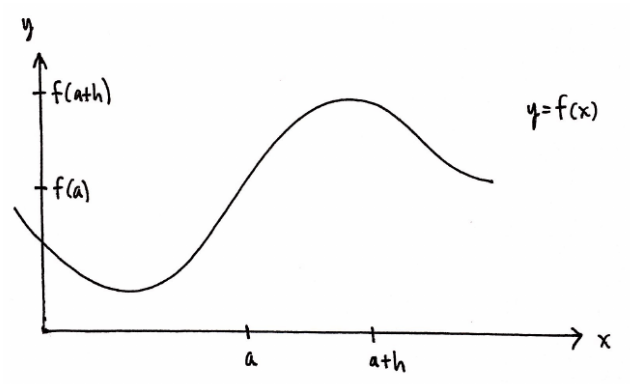
vi.) $\lim_{x \to 3^-} f(x) =$

vii.) $\lim_{x \to 3^+} f(x) =$

viii.) $\lim_{x \to 3} f(x) =$

ix.) At what x-values is the function discontinuous?

8. Consider the following graph of a function $y = f(x)$. Here a is a point in the domain of f and h is a nonzero constant.



a.) Sketch the secant line passing through the points $(a, f(a))$ and $(a + h, f(a + h))$.

b.) Write an expression that represents the slope of this secant line?

c.) Suppose you take the limit as $h \to 0$. Describe what happens to the secant line that you drew in part *a*.

d.) Describe what the limit of the slope in part *b* represents geometrically.

9. Compute $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for the function $f(x) = \sqrt{x}$.

10. Compute $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for the function $f(x) = \frac{1}{x}$.

2. The Derivative

These Good Problems cover section 2.3 of our book.

We begin by reviewing the main conceptual ideas and definitions of the section. The concepts of these first three questions are the most important ideas of the first half of this course.

1. State the limit definition of derivative at a point. Write the limit in both forms.

2. State in words the geometric and physical interpretations of the derivative.

3. Suppose that f is differentiable at $x = a$. Write the general equation of the tangent line to the curve $y = f(x)$ at $x = a$.

4. Consider the function $f(x) = x^3$.

a.) Compute the derivative of f at $x = -1$.

b.) Write an equation of the tangent line to $y = f(x)$ at the point $(-1, -1)$.

5. Compute the derivative of $f(x) = 2x - 1$ at the point $(5, 9)$.

6. Consider the function $f(x) = \sqrt{x}$.

a.) What is the domain of f?

b.) Compute the derivative of f.

c.) What is the domain of f'? Explain in terms of the graph why this is different than your answer to part *a*.

7. Compute the derivative of $f(x) = \dfrac{3}{x}$.

8. Find an equation of the tangent line to the graph of $y = 3x^2 - x$ at the point $(2, 10)$.

9. Find an equation of the tangent line to the graph of $y = (x - 1)^3$ at the point $(-1, -8)$.

10. Find all points on the graph of $y = x^4 - 2x^2$ where the tangent line is horizontal.

Hint: Compute the derivative y', set it equal to 0 (why 0?), then solve for x.

11. Let $s = s(t)$ be a function that represents the position of a particle moving along a straight line. The *velocity* of the particle is defined to be the derivative $v(t) = s'(t)$.

Find the velocity of a particle whose position function is $s(t) = 2t^2 - t$.

3. Derivative Rules

These Good Problems cover section 2.5 of our book.

1. Compute the derivatives of the functions.

1.) $y = 3x^3 - 9x^2 + \ln(x)$

2.) $f(x) = x^7(x+1)^5$

3.) $f(t) = te^t$

4.) $y = \dfrac{x^5}{\ln x}$

5.) $h(t) = \dfrac{4}{t^5} - 4^t$

6.) $q(s) = \log(s)$

7.) $F(x) = \ln(x^2 e^x)$

2. Compute the derivatives of the functions.

1.) $f(x) = (2x^2 - x)^9$

2.) $V(x) = e^{3x^2+5x}$

3.) $g(x) = \dfrac{1}{(x^2-4)^2}$

3. Find an equation of the tangent line to the curve $y = e^{1-x^2}$ at the point $(1, 1)$.

4. Find all points on the curve $y = x - \ln x$ where the tangent line is horizontal.

A manufacturer has determined that an employee with x days of production experience will be able to produce approximately

$$P(x) = 3 + 15(1 - e^{-0.2x})$$

items per day. Use a graphing utility to graph the function.

5. Approximately how many items will a beginning employee be able to produce each day?

6. How many items will an experienced employee (who has worked at the company for years) be able to produce?

7. What is the marginal production rate of an employee with 5 days experience? What are the units and what does this answer mean?

8. An arrow shot straight up from ground level with an initial velocity of 128 feet per second will be at a height

$$h(t) = -16t^2 + 128t$$

feet after t seconds. Determine a.) At what time will the arrow reach its highest point (Hint: the arrow's velocity will be 0 at this point); and b.) How long will the arrow be aloft?

9. Consider a cubic function of the form

$$f(x) = x^3 + Ax^2 + Bx + C.$$

Find conditions on the constants A, B, and C to guarantee that the graph of $y = f(x)$ has two distinct turning points.

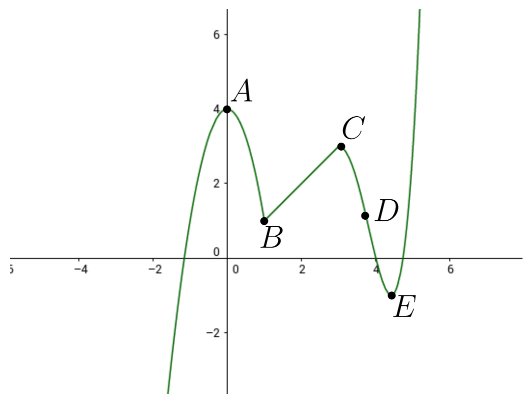
4. Derivative Tests, and Review

These Good Problems study the concepts presented in sections 2.6 and 2.7 of our text, then review material from the entire semester thus far as preparation for the first midterm exam.

1. Find the intervals of concavity of the function $f(x) = 4x^4 - 8x^2 + 5$.

2. Determine the concavity of the function $y = x\ln(x)$ at $x = \frac{1}{e}$.

3. Determine which points on the graph are local extrema, and which are inflection points.



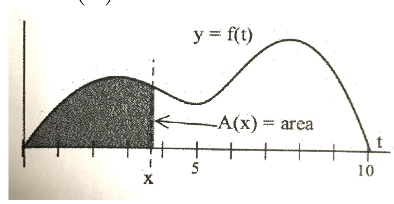
4. Determine the absolute (global) maximum and/or minimum values of the function $f(x) = \dfrac{1}{x^2+1}$, if they exist.

5. Find all critical points and local extreme points of the function

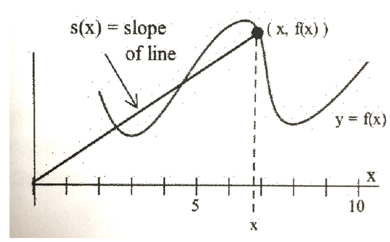
$$f(x) = \ln(x^2 - 6x + 11),$$

and determine whether the extreme points are maxima or minima.

6. let f be a non-negative function on the interval $[0, 10]$ such that $f(0) = f(10) = 0$. Define $A(x)$ to be the area bounded between the x-axis, the graph of $y = f(t)$, and the vertical line at x. At what value of x is $A(x)$ a maximum? At what value of x is $A(x)$ a minimum?



7. Let f be a positive function on the interval $[2, 10]$ with graph shown below. Define $S(x)$ to be the slope of the line through the points $(0, 0$ and $(x, f(x))$. At what value of x is $S(x)$ a maximum? At what value of x is $S(x)$ a minimum?



The remaining questions are intended to be review for the first midterm exam.

8. Compute the limits. Be sure to show enough work and use proper notation.

a.) $\lim_{x \to 4} 3x^2 - 9x + 4$

b.) $\lim_{t \to 0} \frac{t^2 - t}{2t}$

c.) $\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

d.) $\lim_{x \to -2} \frac{x^5 + 32}{x + 2}$

e.) $\lim_{u \to 0^+} \ln u$

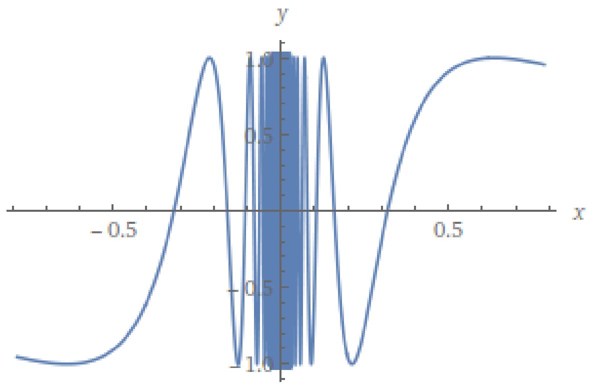
9. Use the *limit definition of derivative* to compute $\frac{dy}{dx}$ for $y = x^2 + 2x$. You **must** use the limit definition and show enough work to receive any credit.

10. Use the *limit definition of derivative* to compute $\frac{dy}{dx}$ for $y = x^6$. You **must** use the limit definition and show enough work to receive any credit.

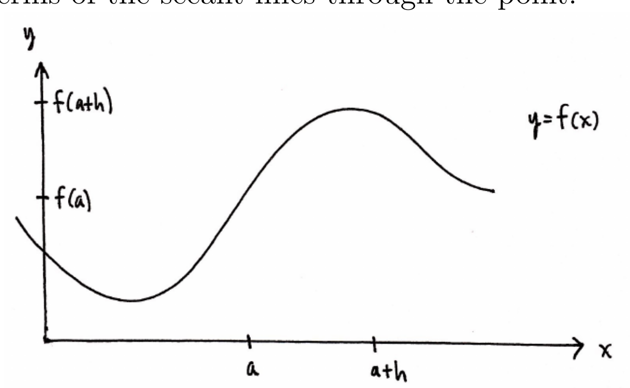
11. Find an equation of the tangent line to the curve $y = \ln(x)$ when $x = 1$.

12. Find all points on the curve $y = e^{6x^3-9x^2}$ at which the tangent line is horizontal.

13. Consider the graph of the function $y = f(x)$ given below. Explain why the limit as x approaches 0 does not exist.



14. Use the graph below to describe how the derivative of a function is defined at a point in terms of the secant lines through the point.



15. Find an equation of the normal line to the graph of $y = x^3$ at the point $(-2, -8)$. (*The normal line is perpendicular to the tangent line.*)

16. Compute the derivatives. You can use the "shortcut" rules, but be sure to show enough work.

a.) $f(x) = 2^x(x^2 - 5x)$

b.) $y = te^{t^2}$

c.) $x(t) = \ln(t^2 - 1)$

d.) $g(x) = \dfrac{\ln x}{x^2 - 1}$

e.) $q(s) = (s^3 - 9)^4$

f.) $p(x) = 6x^5 - \frac{15}{2}x^4 + 10x^3 - 15x^2 + 30x - 3000$

17. State the (limit) definition of continuity.

18. Determine whether the function is continuous. If it is discontinuous, state where.

$$f(x) = \begin{cases} x^2 & x \leq 2, \\ 6 - x & 2 < x < 6, \\ 2x - 17 & x \geq 6. \end{cases}$$

19. Determine whether the function is continuous. If it is discontinuous, state where.

$$g(x) = \frac{12}{5x^3 - 5x}$$

Some Comments

This review is not meant the be comprehensive. You should also study past Good Problems and Recommended Exercises.

The exam will be structured as follows. There will be 5 True/False questions and 5 "Fill in the Blank" questions, each worth 1 point each. Then there will be 18 Multiple Choice questions and 2 Short Answer questions, each worth 5 points each. The Multiple Choice questions are all or nothing (no partial credit), but partial credit will be possible on the Short Answer questions.

You will *not* be allowed to use a calculator or any other electronic device on the exam. You will be allowed to use a single 3×5 in^2 note card of your own hand-written notes. If the note card is too big, or if the notes are not written by hand, then you will not be allowed to use the note card on the exam.

You'll also need to know...

Definitions!

I won't ask you to state any definitions word-for-word, but I will expect you to know them. Definitions are the most important part of this course. Our goal is to eventually use Calculus to help us solve real world problems. But we cannot use Calculus for anything if we don't know what the terms mean.

5. Some Applied Examples

These Good Problems closely follow Section 2.9 of our book. We will apply the tools of Calculus that we have added to our toolbox up to this point in the semester.

1. The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built with redwood fencing at a cost of \$7 per running foot. The fourth side will be built of cement blocks at a cost of \$14 per running foot. Find the dimensions that minimize the cost of the enclosure.

2. A concert promoter has found that if s/he sells tickets for \$50 each then s/he can sell 1200 tickets, but for for every \$5 rise in price 50 less tickets will be sold. What price should the tickets be sold at to maximize revenue?

3. **Thought Exercise**. The profit function is obtained by subtracting the cost function from the revenue function,

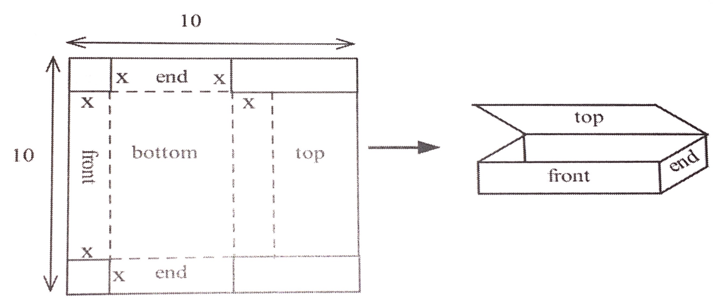
$$P(x) = R(x) - C(x).$$

Determine criteria on the marginal revenue and marginal cost functions in order for the profit function to have a (local) maximum.

4. A company sells q ribbon winders per year at \$$p$ per winder. The demand function for ribbon winders is given by $p = 300 - 0.02q$. The ribbon winders cost \$30 each to manufacture, plus there are fixed costs of \$9000 per year. How many winders should be produced to maximize the profit?
[Reminder: Revenue $= q \cdot p$.]

5. The cost in dollars to produce x gift baskets is given by $C(x) = 160 + 2x + 0.1x^2$. How many baskets should be produced in order to minimize average cost?
[Reminder: Average Cost $= \frac{C(x)}{x}$.]

6. You have a 10 inch by 10 inch piece of cardboard which you plan to cut and fold as shown to form a box with a top. Find the dimensions of the box that has the largest volume.



7. You wish to open a coffee shop. In planning you estimate that if there is enough seating for between 40 and 80 people, the daily profit will be \$50 per seat. However, if the seating capacity is more than 80 seats then the daily profit per seat will be decreased by \$1 per seat over 80. What should the seating capacity be in order to maximize your shop's profit?

6. Related Rates Applications

In these Good Problems we will apply the idea of related rates to solve some real world problems.

For each of these problems, you should begin by drawing a picture (if appropriate), write down any equations that are not given, then do the calculus.

1. A right cylindrical tank is filled with water. The tank stands 2 meters tall and has a radius of 20 cm. How fast does the height of water in the tank drop when the water is being drained at a rate of 25 $\frac{\text{cm}^3}{\text{sec}}$? Find the exact rate that the height is decreasing when the height of the water is 1 meter high.

[Volume of a cylinder: $V = \pi r^2 h$.]

2. A storage bin is in the shape of a right circular cone. The bin stands 10 feet tall and the top rim of the bin has a radius of 4 feet. If a grain is being pumped into the bin at a rate of 2 cubic feet per minute, what is the rate of change of the depth of the grain when the depth is 5 feet?

[Volume of a cone: $V = \frac{1}{3}\pi r^2 h$.]

3. A company that manufactures pet toys calculates that its cost and revenue can be modeled by the functions

$$C(x) = 50000 + 3.1x$$
$$R(x) = 300x - 0.2x^2$$

where x represents the number of toys produced in a week. If production in one particular week is increasing at a rate of 300 toys per week, find:

a.) the rate at which the cost is changing,

b.) the rate at which the revenue is changing,

c.) the rate at which the profit is changing.

4. A company's profit is increasing at a rate of \$6000 per week. The demand and cost functions are given by

$$p(x) = 5000 - 30x,$$
$$C(x) = 3000x + 6000.$$

Find the rate of change of sales $\left(\frac{dx}{dt}\right)$ when the weekly sales are $x = 50$ units.

5. A young woman and her boyfriend plan to elope, but she must first rescue him from his mother who has locked him in his room. The woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second. How fast is the top of the ladder (including the young couple!) falling when the bottom of the ladder is

a.) 12 feet from the base of the wall?

b.) 16 feet from the base of the wall?

c.) 19 feet from the base of the wall?

6. Two resistors are connected in parallel with resistances R_1 and R_2 measured in Ohms (Ω). The total resistance, R, is given implicitly by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Suppose that R_1 is increasing at a rate of $0.4\,\frac{\Omega}{\text{min}}$ and R_2 is decreasing at a rate of $0.7\,\frac{\Omega}{\text{min}}$. At what rate is R changing when $R_1 = 80\,\Omega$ and $R_2 = 105\,\Omega$?

7. Linear Approximations and Differentials

This week's Good Problems cover the linearizations of smooth functions. This topic is not in our book, but you can find my notes online at http://geometerjustin.com/teaching/bc/notes/.

1. Find the differential of each function.

a.) $y = x^2 e^x$

b.) $y = \dfrac{x}{1+2x}$

c.) $y = \sqrt{4+5x}$

d.) $y = \dfrac{1}{1+x}$

2. Find the linearization $y = L_a f(x)$ of the given function at the given value of a.

a.) $f(x) = x^4 + 3x^2, \quad a = -1$

b.) $f(x) = \dfrac{1}{\sqrt{2+x}}, \quad a = 0$

c.) $f(x) = x^{3/4}, \quad a = 16$

3. Use a linear approximation or differentials to approximate $(2.001)^5$.

4. Use a linear approximation or differentials to approximate $\dfrac{1}{1002}$.

5. Verify that the linearization of $f(x) = \sqrt[3]{1-x}$ at $a = 0$ is given by $L_0 f(x) = 1 - \frac{1}{3}x$.

For what values of x is $L_0 f$ accurate to within 0.1?

6. Consider the function $y = \sqrt{x}$. Compare the values of dy and Δy when $x = 1$ and $dx = \Delta x = 1$. Sketch a diagram of two triangles with segment lengths dx, dy, and Δy.

7. The edge of a cube is measured to be 30 cm, with a possible error of 0.1 cm. Use differentials to estimate the maximum possible error and relative error in computing the surface area of the cube.

8. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm. Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

9. When blood flows along a blood vessel, *Poiseuille's Law* states that the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4.$$

A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery to widen it and restore the normal blood flow.

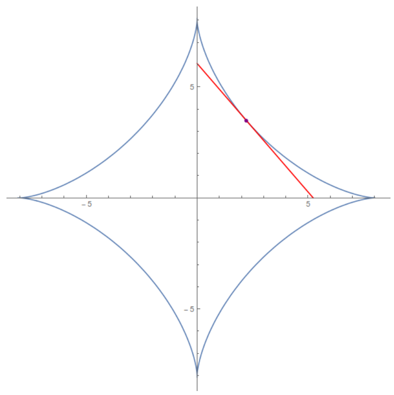
Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

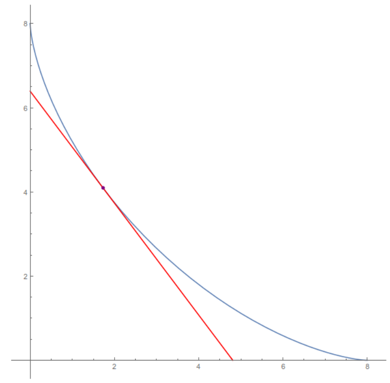
10. Suppose that you don't have a formula for f, but you know that $f(2) = -4$ and $f'(x) = \sqrt{x^2 + 5}$ for all x.

Use a linear approximation to estimate $f(1.95)$ and $f(2.05)$.

Are your estimates too large or too small? Explain.

11. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = 4$ cut off by the coordinate axes is constant.





You can also find a link on my web page to a version of this graph with a slider to move the point of tangency along the curve:
https://geometerjustin.com/teaching/bc/gp/