1. Limits

These Good Problems cover material from section 2.2 of our book. These good problems consist of a mix of computational and conceptual exercises. The goal is to cement the ideas of this week's lecture, with an eye looking forward to next week's lecture.

1. Consider the function $f(x) = \dfrac{x^2+3x+3}{x-2}$. Compute the following limits, if they exist.

a.) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2+3x+3}{x-2} = \dfrac{1^2+3(1)+3}{1-2} = \dfrac{7}{-1} = -7$

b.) $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2+3x+3}{x-2} = \dfrac{2^2+3(2)+3}{2-2} = \dfrac{13}{0} = \text{DNE}$

2. Consider the function $f(x) = \dfrac{2x^2-x-1}{x-1}$.

a.) What is the domain of f?

$x-1 \neq 0$

$x \neq 1$

b.) Calculate $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{2x^2-x-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(2x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim\limits_{x\to 1} 2x+1$

$= 2(1)+1$

$= 3$

3. Consider the function $f(x) = \dfrac{x+7}{x^2+9x+14} = \dfrac{x+7}{(x+7)(x+2)}$

a.) Compute $\lim_{x\to -7} f(x)$

$$= \lim_{x\to -7} \frac{\cancel{x+7}}{\cancel{(x+7)}(x-2)} = \lim_{x\to -7} \frac{1}{x-2} = \frac{1}{-7-2} = -\frac{1}{9}$$

b.) Compute $\lim_{x\to -2} f(x)$

$$= \lim_{x\to -2} \frac{\cancel{x+7}}{\cancel{(x+7)}(x+2)} = \lim_{x\to -2} \frac{1}{x-2} = \frac{1}{0} = \text{DNE}$$

Use a graphing utility to graph the function $y = f(x)$. Explain in terms of the graph of $y = f(x)$ why the limits $\lim_{x\to -7} f(x)$ and $\lim_{x\to -2} f(x)$ are fundamentally different.

The graph of $y=f(x)$ has a hole at $x=-7$ and a vertical asymptote at $x=-2$.

At a hole, the limit exists.

At an asymptote, it does not.

4. Consider the function $f(x) = x^2$.

a.) What is $f(2)$?

$f(2) = 2^2 = 4$

b.) Compute $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ $= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2} x+2$

$= 2+2 = 4$

5. Consider the function $f(x) = x^6$.

a.) Compute the difference quotient of f, $DQ = \dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$. [Hint: Use Pascal's Triangle (Binomial Theorem) to compute $f(x+h)$.]

$f(x+h) = (x+h)^6$

$= \cancel{x^6} + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6$

$- f(x) = \cancel{x^6}$

$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{h}(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)}{\cancel{h}}$

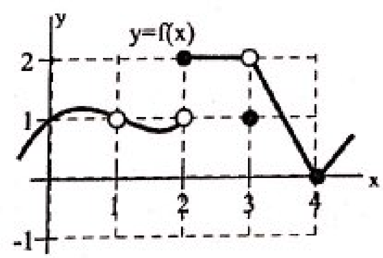
$DQ = 6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5$

b.) Compute $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

$\lim_{h \to 0} (6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)$

$= 6x^5$

6. Consider the following graph.



Compute the following,

i.) $f(1) =$ undef.

ii.) $\lim_{x \to 1} f(x) =$ 1

iii.) $f(3) =$ 1

iv.) $\lim_{x \to 3} f(x) =$ 2

v.) $f(2) =$ 2

vi.) $\lim_{x \to 2^-} f(x) =$ 1

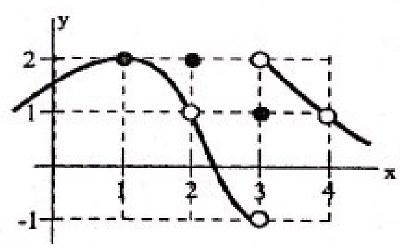
vii.) $\lim_{x \to 2^+} f(x) =$ 2

viii.) $\lim_{x \to 2} f(x) =$ DNE

ix.) At what x-values is the function discontinuous?

x=1,2,3

7. Consider the following graph.



Compute the following,

i.) $f(2) =$ 2

ii.) $\lim_{x \to 2} f(x) =$ 1

iii.) $f(4) =$ undef.

iv.) $\lim_{x \to 4} f(x) =$ 1

v.) $f(3) =$ 1

vi.) $\lim_{x \to 3^-} f(x) =$ -1

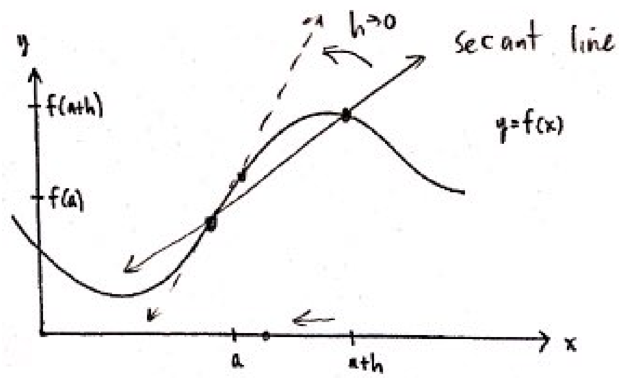
vii.) $\lim_{x \to 3^+} f(x) =$ 2

viii.) $\lim_{x \to 3} f(x) =$ DNE

ix.) At what x-values is the function discontinuous?

x= 2, 3,4

8. Consider the following graph of a function $y = f(x)$. Here a is a point in the domain of f and h is a nonzero constant.



a.) Sketch the secant line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$.

b.) Write an expression that represents the slope of this secant line?

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - a} = \boxed{\frac{f(a+h) - f(a)}{h}}$$

c.) Suppose you take the limit as $h \to 0$. Describe what happens to the secant line that you drew in part *a*.

As $h \to 0$, $(a+h) \to a$.

The secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$ will approach the tangent line to the graph at $(a, f(a))$.

d.) Describe what the limit of the slope in part *b* represents geometrically.

Based on the answer to part c.),

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

should represent the slope of the tangent line to the graph at $(a, f(a))$.

9. Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for the function $f(x)=\sqrt{x}$.

$$f(x+h) = \sqrt{x+h}$$
$$-\ f(x) = \sqrt{x}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\ (\sqrt{x+h}+\sqrt{x})} = \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

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10. Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for the function $f(x)=\frac{1}{x}$.

$$f(x+h) = \frac{1}{x+h}$$
$$-\ f(x) = \frac{1}{x}$$

$$f(x+h)-f(x) = \frac{1\cdot x}{x\ (x+h)} - \frac{1\cdot(x+h)}{x\ (x+h)} = \frac{x-(x+h)}{x\ (x+h)} = \frac{-h}{x\ (x+h)}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{-h}{h\,x\ (x+h)}$$

$$\lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}\,x\,(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x\ (x)} = \frac{-1}{x^2}$$