2. The Derivative

These Good Problems cover section 2.3 of our book.

We begin by reviewing the main conceptual ideas and definitions of the section. The concepts of these first three questions are the most important ideas of the first half of this course.

1. State the limit definition of derivative at a point. Write the limit in both forms.

The derivative of $y = f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists

2. State in words the geometric and physical interpretations of the derivative.

Geometric: The derivative is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

Physical: The derivative is the instantaneous rate of change of the function $y = f(x)$ when $x = a$.

3. Suppose that f is differentiable at $x = a$. Write the general equation of the tangent line to the curve $y = f(x)$ at $x = a$.

$m = f'(a)$ by definition.

Pt-Slope formula: $y - f(a) = f'(a)(x - a)$

or

$$\boxed{y = f(a) + f'(a)(x - a)}$$

4. Consider the function $f(x) = x^3$.

a.) Compute the derivative of f at $x = -1$.

$f(6) = (-1)^3 = -1$

$$f'(-1) = \lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}} = \lim_{x\to -1} x^2-x+1$$

$$= (-1)^2-(-1)+1 = 1+1+1 = 3.$$

$$\boxed{f'(-1)=3}$$

b.) Write an equation of the tangent line to $y = f(x)$ at the point $(-1, -1)$.

$$y = -1+3(x+1)$$
$$= -1+3x+3$$
$$\boxed{y = 3x+2}$$

5. Compute the derivative of $f(x) = 2x - 1$ at the point $(5, 9)$.

$f(5) = 9$

$$f'(5) = \lim_{x\to 5} \frac{(2x-1)-9}{x-5} = \lim_{x\to 5} \frac{2x-10}{x-5} = \lim_{x\to 5} \frac{2\cancel{(x-5)}}{\cancel{x-5}}$$

$$= \lim_{x\to 5} 2 = 2.$$

So, $\boxed{f'(5)=2}$

6. Consider the function $f(x) = \sqrt{x}$.

a.) What is the domain of f?

$$\mathrm{dom}(f) = [0,\infty) \quad \text{or} \quad x \geq 0.$$

b.) Compute the derivative of f.

$$f'(x) = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h}\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} - \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}-\sqrt{x})} = \lim_{h\to 0} \frac{\cancel{h}\, 1}{\cancel{h}(\sqrt{x+h}-\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\text{so} \quad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

c.) What is the domain of f'? Explain in terms of the graph why this is different than your answer to part *a*.

$\mathrm{dom}(f') = (0,\infty)$ or $x > 0$. $x = 0$ is no longer allowed.
This is because $y = \sqrt{x}$ has a vertical tangent line at $x = 0$.

7. Compute the derivative of $f(x) = \dfrac{3}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{3}{x+h} - \frac{3}{x}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{3x - 3(x+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{3x - 3x - 3h}{h\cdot x(x+h)} = \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}\cdot x(x+h)} = \lim_{h\to 0} \frac{-3}{x(x+h)}$$

$$= \frac{-3}{x^2}$$

$$\boxed{\left(\frac{3}{x}\right)' = \frac{-3}{x^2}}$$

8. Find an equation of the tangent line to the graph of $y = 3x^2 - x$ at the point $(2, 10)$.

$$y' = \lim_{h\to 0} \frac{3(x+h)^2 - (x+h) - (3x^2 - x)}{h} = \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 - h}{h} = \lim_{h\to 0} \frac{\cancel{h}(6x + 3h - 1)}{\cancel{h}} = \lim_{h\to 0} (6x + 3h - 1)$$

$$= 6x - 1$$

So $f'(2) = 6(2) - 1 = 12 - 1 = 11$

Then $y = 10 + 11(x-2)$
$= 10 + 11x - 22$
$= 11x - 12$

So $\boxed{y = 11x - 12}$ is the eqn of the tan. line.

9. Find an equation of the tangent line to the graph of $y = (x-1)^3$ at the point $(-1, -8)$.

$$y = x^3 - 3x^2 + 3x - 1$$

$$f'(-1) = \lim_{x\to -1} \frac{x^3 - 3x^2 + 3x - 1 + 8}{x+1} = \lim_{x\to -1} \frac{x^3 - 3x^2 + 3x + 7}{x+1}$$

Synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 3 & 7 \\ & \downarrow & -1 & 4 & -7 \\ \hline & 1 & -4 & 7 & \boxed{0} \end{array}$$

$$= \lim_{x\to -1} (x^2 - 4x + 7) = (-1)^2 - 4(-1) + 7 = 1 + 4 + 7 = 12.$$

So $y = -8 + 12(x+1)$
$= -8 + 12x + 12$
$= 12x + 4$

$\boxed{y = 12x + 4}$

10. Find all points on the graph of $y = x^4 - 2x^2$ where the tangent line is horizontal.

Hint: Compute the derivative y', set it equal to 0 (why 0?), then solve for x.

$$f'(a) = \lim_{x \to a} \frac{x^4 - 2x^2 - a^4 + 2a^2}{x - a}$$

Synth. div.:

$$\begin{array}{r|rrrrr} a & 1 & 0 & -2 & 0 & -a^4+2a^2 \\ & \downarrow & a & a^2 & -2a+a^3 & -2a^2+a^4 \\ \hline & 1 & a & -2+a^2 & -2a+a^3 & \boxed{0} \end{array}$$

$$= \lim_{x \to a} \left(x^3 + ax^2 - 2x + a^2 x - 2a + a^3 \right)$$

$$= a^3 + a^3 - 2a + a^3 - 2a + a^3$$

$$= 4a^3 - 4a$$

so $f'(x) = 4x^3 - 4x$

$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$4x(x+1)(x-1) = 0$$

$$\boxed{x = 0, \pm 1}$$

The pts are:

(0, 0)

(1, -1)

(-1, -1)

11. Let $s = s(t)$ be a function that represents the position of a particle moving along a straight line. The *velocity* of the particle is defined to be the derivative $v(t) = s'(t)$.

Find the velocity of a particle whose position function is $s(t) = 2t^2 - t$.

$$s'(t) = \lim_{h \to 0} \frac{2(t+h)^2 - (t+h) - 2t^2 + t}{h} = \lim_{h \to 0} \frac{\cancel{2t^2} + 4th + 2h^2 - \cancel{t} - h - \cancel{2t^2} + \cancel{t}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4t - 1 + 2h)}{\cancel{h}} = \lim_{h \to 0} 4t - 1 + 2h = 4t - 1$$

so $\boxed{v(t) = 4t - 1}$