4. Derivative Tests, and Review

These Good Problems study the concepts presented in sections 2.6 and 2.7 of our text, then review material from the entire semester thus far as preparation for the first midterm exam.

1. Find the intervals of concavity of the function $f(x) = 4x^4 - 8x^2 + 5$.

$f(x) = 4x^4 - 8x^2 + 5$

$f'(x) = 16x^3 - 16x$

$f''(x) = 48x^2 - 16$

$48x^2 - 16 = 0$

$16(3x^2 - 1) = 0$

$3x^2 = 1$

$x = \pm\frac{1}{\sqrt{3}}$

$\cup \quad -\frac{1}{\sqrt{3}} \quad \cap \quad \frac{1}{\sqrt{3}} \quad \cup \qquad x,\ f''$

Test Value	Sign of f''
-1	$+$
0	$-$
1	$+$

Thus, f is $\begin{cases} \text{concave up on } (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty) \\ \text{concave down on } (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}). \end{cases}$

2. Determine the concavity of the function $y = x\ln(x)$ at $x = \frac{1}{e}$.

$y = x\ln(x)$

$y' = \ln(x) + x\left(\frac{1}{x}\right)$

$= \ln(x) + 1$

$y'' = \frac{1}{x}$

$y''\left(\frac{1}{e}\right) = \frac{1}{1/e} = e > 0$

So the graph of y is concave up at $x = \frac{1}{e}$.

3. Determine which points on the graph are local extrema, and which are inflection points.

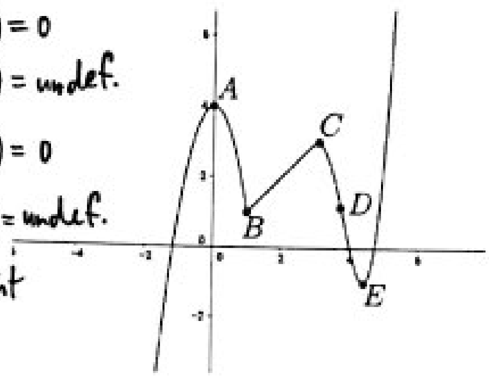
A: local max, $f'(x) = 0$

C: local max, $f'(x) =$ undef.

E: local min, $f'(x) = 0$

B: local min, $f'(x) =$ undef.

D: inflection point



4. Determine the absolute (global) maximum and/or minimum values of the function $f(x) = \dfrac{1}{x^2+1}$, if they exist.

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$ by the chain rule.

<u>Critical</u> <u>Numbers</u>:

$-2x = 0 \Rightarrow \boxed{x=0}$

$(x^2+1)^2 = 0 \Rightarrow x^2+1 = 0$

$\Rightarrow$ NONE

To determine if $x=0$ corresponds to a max. or min., take the second derivative:

$$f''(x) = \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{\cancel{(x^2+1)}(8x^2-2x^2-2)}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{2(3x^2-1)}{(x^2+1)^3}$$

$f''(0) = \frac{-2}{1} < 0$, so the graph is concave down at $x=0$.

Hence $x=0$ is a <u>max</u>.

The <u>value</u> of f at the max point is $f(0) = \frac{1}{0^2+1} = 1$. Thus, the point $(0,1)$ is the <u>global</u> <u>max</u> of the function.

5. Find all critical points and local extreme points of the function

$$f(x) = \ln(x^2 - 6x + 11),$$

and determine whether the extreme points are maxima or minima.

$$f'(x) = \frac{2x-6}{x^2-6x+11}$$

$$f''(x) = \frac{2(x^2-6x+11) - (2x-6)^2}{(x^2-6x+11)^2}$$

$$= \frac{2x^2-12x+22-4x^2+24x-36}{(x^2-6x+11)^2}$$

$$= \frac{-2x^2+12x-14}{(x^2-6x+11)^2}$$

$$= \frac{-2(x^2-6x+7)}{(x^2-6x+11)^2}$$

Critical numbers:

$2x-6=0 \Rightarrow x=3.$

$x^2-6x+11=0 \Rightarrow$ No real soln.

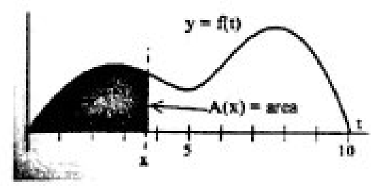
So the only critical number is $x=3$.

The critical point is $(3, \ln 2)$.

$f''(3) = \frac{4}{4} = 1 > 0$

So the point $(3, \ln 2)$ is a local minimum.

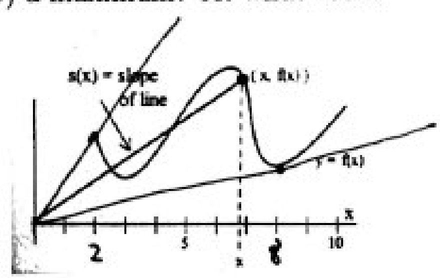
6. let f be a non-negative function on the interval $[0, 10]$ such that $f(0) = f(10) = 0$. Define $A(x)$ to be the area bounded between the x-axis, the graph of $y = f(t)$, and the vertical line at x. At what value of x is $A(x)$ a maximum? At what value of x is $A(x)$ a minimum?



$A(x)$ is max @ $x = 10$

$A(x)$ is min @ $x = 0$

7. Let f be a positive function on the interval $[2, 10]$ with graph shown below. Define $S(x)$ to be the slope of the line through the points (0, 0 and $(x, f(x))$. At what value of x is $S(x)$ a maximum? At what value of x is $S(x)$ a minimum?



$S(x)$ is max @ $x = 2$

$S(x)$ is min @ $x = 8$

The remaining questions are intended to be review for the first midterm exam.

8. Compute the limits. Be sure to show enough work and use proper notation.

a.) $\lim\limits_{x\to 4} 3x^2 - 9x + 4$ $= 3(4)^2 - 9(4) + 4$

$= 48 - 36 + 4$

$= 16$

b.) $\lim\limits_{t\to 0} \dfrac{t^2 - t}{2t} = \lim\limits_{t\to 0} \dfrac{\cancel{t}(t-1)}{2\cancel{t}} = \lim\limits_{t\to 0} \dfrac{t-1}{2} = \dfrac{0-1}{2} = \dfrac{-1}{2}$

c.) $\lim\limits_{x\to a} \dfrac{(\sqrt{x} - \sqrt{a})}{(x - a)} \dfrac{(\sqrt{x}+\sqrt{a})}{(\sqrt{x}+\sqrt{a})} = \lim\limits_{x\to a} \dfrac{(\sqrt{x})^2 - (\sqrt{a})^2}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim\limits_{x\to a} \dfrac{\cancel{(x-a)}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$

$= \lim\limits_{x\to a} \dfrac{1}{\sqrt{x}+\sqrt{a}} = \dfrac{1}{\sqrt{a}+\sqrt{a}} = \dfrac{1}{2\sqrt{a}}$

d.) $\lim\limits_{x\to -2} \dfrac{x^5 + 32}{x + 2} = \lim\limits_{x\to -2} (x^4 - 2x^3 + 4x^2 - 8x + 16) = (-2)^4 - 2(-2)^3 + 4(-2)^2 - 8(-2) + 16$

Synthetic Division:

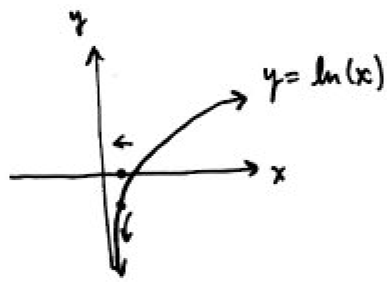
$$\begin{array}{r|rrrrrr} -2 & 1 & 0 & 0 & 0 & 0 & 32 \\ & & -2 & 4 & -8 & 16 & -32 \\ \hline & 1 & -2 & 4 & -8 & 16 & \boxed{0} \end{array}$$

$= 16 + 16 + 16 + 16 + 16$

$= 5(16) = 80.$

e.) $\lim\limits_{u\to 0^+} \ln u$

$\lim\limits_{u\to 0^+} \ln(u) = -\infty.$



9. Use the *limit definition of derivative* to compute $\frac{dy}{dx}$ for $y = x^2 + 2x$. You **must** use the limit definition and show enough work to receive any credit.

$$\frac{dy}{dx} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^2+2x-a^2-2a}{x-a} = \lim_{x \to a} \frac{x^2-a^2}{x-a} + \lim_{x \to a} \frac{2x-2a}{x-a}$$

$$= \lim_{x \to a} \frac{(x+a)\cancel{(x-a)}}{\cancel{x-a}} + \lim_{x \to a} \frac{2\cancel{(x-a)}}{\cancel{x-a}}$$

$$= \lim_{x \to a} (x+a) + \lim_{x \to a} 2$$

$$= (a+a) + 2$$

$$= 2a + 2$$

Replacing a w/ x, we obtain

$$\boxed{\frac{dy}{dx} = 2x+2}$$

10. Use the *limit definition of derivative* to compute $\frac{dy}{dx}$ for $y = x^6$. You **must** use the limit definition and show enough work to receive any credit.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$f(x+h) = (x+h)^6 = \cancel{x^6} + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6 \quad \text{(Pascal's } \Delta)$$

$$-\ f(x) = \cancel{x^6}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{h(6x^5+15x^4h+20x^3h^2+15x^2h^3+6xh^4+h^5)}{h}$$

$$\text{So, } \frac{dy}{dx} = \lim_{h \to 0} \frac{\cancel{h}(6x^5+15x^4h+20x^3h^2+15x^2h^3+6xh^4+h^5)}{\cancel{h}}$$

$$= 6x^5 + 15x^4(0) + 20x^3(0)^2 + 15x^2(0)^3 + 6x(0)^4 + 0^5$$

$$= 6x^5$$

$$\text{So, } \boxed{\frac{dy}{dx} = 6x^5}$$

11. Find an equation of the tangent line to the curve $y = \ln(x)$ when $x = 1$.

$y' = \frac{1}{x}$ $y'(1) = \frac{1}{1} = 1.$ $y(1) = \ln(1) = 0$

Tangent Line: $y - 0 = 1(x-1)$

or

$\boxed{y = x-1}$

12. Find all points on the curve $y = e^{6x^3-9x^2}$ at which the tangent line is horizontal.

$y' = (18x^2 - 18x)\, e^{6x^3-9x^2}$

This can equal zero only if

$18x(x-1) = 0$

$x=0$ $x=1$

The points on the curve are:

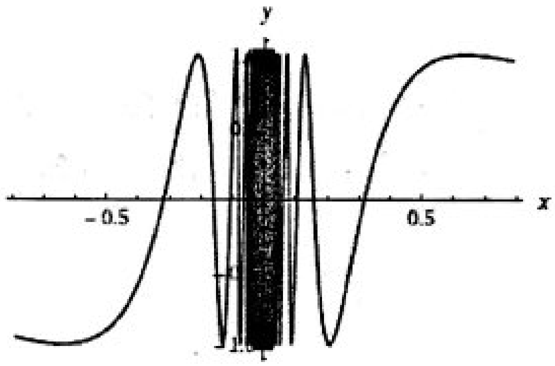
$y(0) = e^0 = 1$

and $y(1) = e^{6-9} = e^{-3}$

$\boxed{(0,1) \text{ and } (1, e^{-3})}$

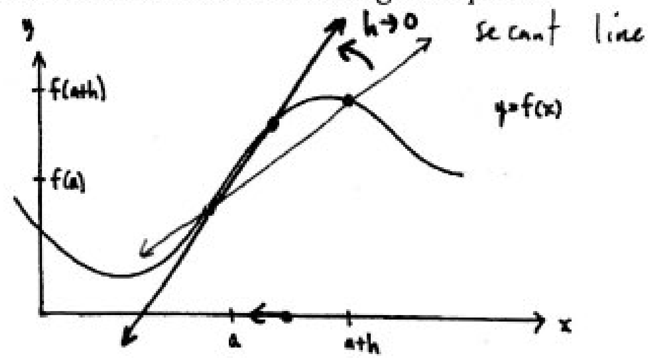
13. Consider the graph of the function $y = f(x)$ given below. Explain why the limit as x approaches 0 does not exist.

The graph does not approach a single height as $x \to 0$. It oscellates between $y=1$ and $y=-1$.



14. Use the graph below to describe how the derivative of a function is defined at a point in terms of the secant lines through the point.

The derivative of f at $x=a$ is the limit of the slope of the secant line as h approaches 0, so $a+h \to a$.



15. Find an equation of the normal line to the graph of $y = x^3$ at the point $(-2, -8)$. (*The normal line is perpendicular to the tangent line.*)

$y' = 3x^2 \qquad y'(-2) = 3(-2)^2 = 12$

The slope of the normal line is therefore $m = \frac{-1}{12}$.

The equation of the normal line is

$$y + 8 = \frac{-1}{12}(x+2)$$

$$y = \frac{-1}{12}x - \frac{2}{12} - 8$$

$$\boxed{y = \frac{-1}{12}x \ \frac{-49}{6}}$$

16. Compute the derivatives. You can use the "shortcut" rules, but be sure to show enough work.

a.) $f(x) = 2^x(x^2 - 5x)$

$f'(x) = \ln(2) \cdot 2^x (x^2-5x) + 2^x(2x-5)$

$\boxed{f'(x) = 2^x\left(\ln(2)\cdot(x^2-5x) + 2x-5\right)}$

b.) $y = te^{t^2}$

$y' = e^{t^2} + 2t^2e^{t^2}$

$\boxed{y' = e^{t^2}(1+2t^2)}$

c.) $x(t) = \ln(t^2 - 1)$

$f = \ln u \quad u = t^2-1$

$f' = \frac{1}{u} \quad u' = 2t$

$x'(t) = 2t/u$

$\boxed{x'(t) = \frac{2t}{t^2-1}}$

d.) $g(x) = \dfrac{\ln x}{x^2 - 1}$

$g'(x) = \dfrac{(x^2-1)(1/x) - 2x\cdot\ln(x)}{(x^2-1)^2}$

e.) $q(s) = (s^3 - 9)^4$

$q'(s) = 12s^2(s^3-9)^3$

f.) $p(x) = 6x^5 - \frac{15}{2}x^4 + 10x^3 - 15x^2 + 30x - 3000$

$p'(x) = 30x^4 - 30x^3 + 30x^2 - 30x + 30$

$\boxed{p'(x) = 30\left(x^4 - x^3 + x^2 - x + 1\right)}$

17. State the (limit) definition of continuity.

A function f is <u>continuous</u> at $x=a$ if and only if

$$\lim_{x\to a} f(x) = f(a).$$

(Note: this implicitly assumes that $f(a)$ is defined and that $\lim_{x\to a} f(x)$ exists.)

18. Determine whether the function is continuous. If it is discontinuous, state where.

$$f(x) = \begin{cases} x^2 & x \le 2, \\ 6-x & 2 < x < 6, \\ 2x-17 & x \ge 6. \end{cases}$$

$$\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} x^2 = 2^2 = 4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} 6-x = 6-2 = 4 \end{aligned}\right\} \text{Continuous at } x=2.$$

$$\left.\begin{aligned} \lim_{x\to 6^-} f(x) &= \lim_{x\to 6^-} 6-x = 6-6 = 0 \\ \lim_{x\to 6^+} f(x) &= \lim_{x\to 6^+} 2x-17 = 12-17 = -5 \end{aligned}\right\} \underline{\text{Not}} \text{ continuous at } x=6.$$

19. Determine whether the function is continuous. If it is discontinuous, state where.

$$g(x) = \frac{12}{5x^3 - 5x}$$

The function is not continuous when the denominator is 0:

$$5x^3 - 5x = 0 \Rightarrow 5x(x^2-1) = 0$$

$$x = 0, \quad x = \pm 1$$

So, the function is discontinuous at $x=0$, $x=-1$, and $x=1$.