Name: Key
M242: Calculus I (Fall 2018)
Instructor: Justin Ryan
Final Exam

Read and follow all instructions. You may not use any electronic devices. You may use a single two-sided 8.5 by 11 inch page of your own hand-written notes.

Part I: True/False [2 points each]
*Neatly write **T** if the statement is always true, and **F** otherwise.*

F **1.** Let f be a function satisfying $f(a) = k$. Then $\lim_{x\to a} f(x) = k$.

f must be continuous at $x=a$ for this to be true.

T **2.** If f is differentiable at $x = a$, then f is continuous at $x = a$.

F **3.** Suppose f' exists. The domain of f' coincides with the domain of f.

Counterexample: $y=\sqrt{x}$, but $y' = \frac{1}{2\sqrt{x}}$

F **4.** If f and g are increasing on (a, b), then fg is increasing on (a, b).

$(fg)' = f'g + fg'$ could be less than 0. Ex. if $f<0$ and $|fg'| > |f'g|$.

F **5.** If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.

f'' must change signs at $x=2$.

T **6.** If f and g are continuous on $[a, b]$, then $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

T **7.** Every continuous function has a continuous antiderivative.

FTC!

T **8.** Let f be a continuous function on the interval $[a, b]$. Then there exists a number c in $[a, b]$ such that $f(c) = f_{\text{avg}}$.

MVT!

T **9.** If f is a continuous function on $[a, a+h]$, then $\lim_{h\to 0^+} f_{\text{avg}} = f(a)$.

by MVT

F **10.** $\frac{d}{dx}[10^x] = x10^{x-1}$

$= \ln(10)\cdot 10^x$

Part II: Computations [5 points each]

Compute the following limits, derivatives, and integrals. Show enough work. Partial credit will be given when deserved.

11. Compute $\lim_{x\to 3} \frac{x^2+2x-5}{x+2}$. $= \frac{9+6-5}{5} = \frac{10}{5} = \boxed{2.}$

12. Compute $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$. $= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = \lim_{x\to 2}(x+3) = \boxed{5}$

13. Compute the derivative of $g(x) = x^2 \sin(\pi x)$.

$$\boxed{g'(x) = 2x\sin(\pi x) + \pi x^2 \cos(\pi x)}$$

14. Compute the derivative of $f(t) = \frac{t^4-1}{t^4+1}$. $= \frac{t^4+1}{t^4+1} - \frac{2}{t^4+1} = 1 - \frac{2}{t^4+1}$

So $f'(t) = \frac{2\cdot 4t^3}{(t^4+1)^2} = \boxed{\frac{8t^3}{(t^4+1)^2}}$

15. Compute the derivative of $f(x) = x^2\sqrt{x^2+1}$.

$$f'(x) = 2x\sqrt{x^2+1} + \frac{x^2 \cdot 2x}{2\sqrt{x^2+1}} = \frac{2x^3+2x+x^3}{\sqrt{x^2+1}}$$

$$= \boxed{\frac{3x^3+2x}{\sqrt{x^2+1}}}$$

16. Find the particular antiderivative of $f(x) = \sin x + x$ satisfying $F(0) = 2$.

$$F(x) = \cos x + \frac{1}{2}x^2 + C$$

$$F(0) = 1 + 0 + C = 2 \implies C = 1$$

So, $\boxed{F(x) = \cos(x) + \frac{1}{2}x^2 + 1}$

17. Compute the integral $\int_0^1 \frac{(\arcsin(x))^3}{\sqrt{1-x^2}}\,dx$.

$$\left\{\begin{array}{ll} u = \arcsin(x) & u(1) = \pi/2 \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & u(0) = 0 \end{array}\right\}$$

$$= \int_0^{\pi/2} u^3\,du = \frac{1}{4}u^4\Big|_0^{\pi/2} = \frac{\pi^4}{2^6} = \boxed{\frac{\pi^4}{64}}$$

18. Compute the integral $\frac{1}{2}\int \frac{2x}{x^2+1}\,dx$.

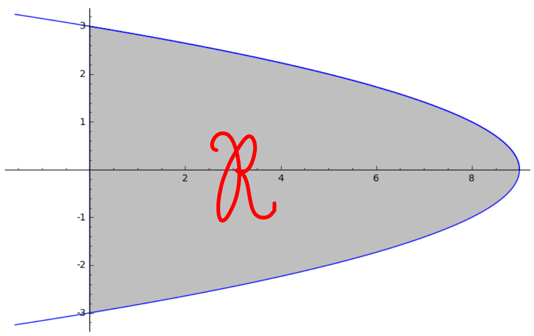
$u = x^2+1$

$du = 2x\,dx$

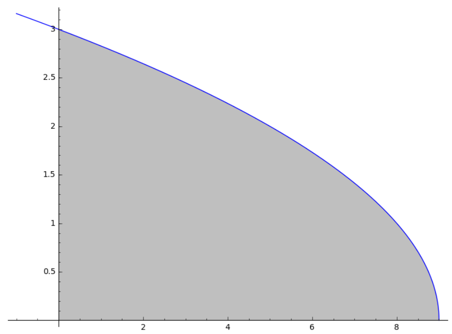
$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C = \boxed{\ln\sqrt{x^2+1} + C}$$

19–21. Consider the region $\mathscr{R}$ bounded between the curves $x = 0$ and $x = 9 - y^2$, and the solid region $\mathscr{S}$ obtained by rotating $\mathscr{R}$ about the line $x = -1$.

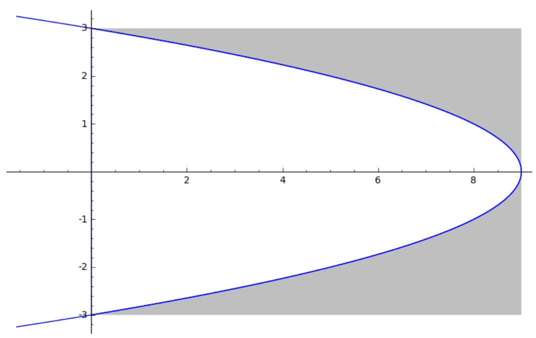
A **19.** Which graph best represents the region $\mathscr{R}$?



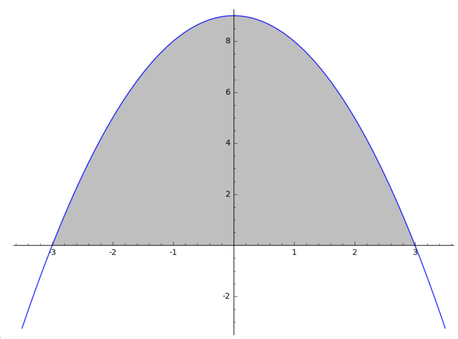
A.



B.



C.



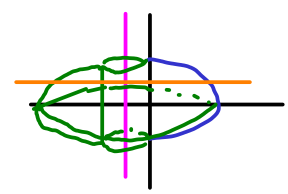
D.

B **20.** Which integral represents the volume of the solid $\mathscr{S}$ using the slicing method?

A. $\pi \int_0^9 9 - x\, dx$ **B.** $\pi \int_{-3}^{3} \left((10 - y^2)^2 - 1 \right)\, dy$

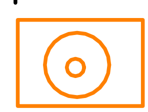
R^2 r^2

C. $\int_0^9 2\pi x \sqrt{9 - x}\, dx$ **D.** $2 \int_0^9 2\pi (x+1) \sqrt{9 - x}\, dx$



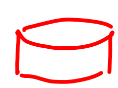
$A = \pi (R^2 - r^2)$

$R = 1 + 9 - y^2$ $r = 0 + 1$



C **21.** Which integral represents the volume of the solid $\mathscr{S}$ using the method of cylindrical shells? (Hint: Use symmetry.)

$r = x + 1$



$h = 2y$

$y = \sqrt{9 - x}$

A. $\pi \int_{-3}^{3} \left((10 - y^2)^2 - 1 \right)\, dy$ **B.** $\pi \int_0^9 9 - x\, dx$

C. $2 \int_0^9 2\pi (x+1) \sqrt{9 - x}\, dx$ **D.** $\int_0^9 2\pi x \sqrt{9 - x}\, dx$

R H

22. Find y' if $xe^y = y - 1$.

$$\frac{d}{dx}\left[xe^y = y-1\right] \rightarrow \quad e^y + xe^y\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(xe^y - 1\right) = -e^y$$

$$\boxed{\frac{dy}{dx} = \frac{e^y}{1-xe^y}}$$

23. Use the limit definition of derivative to compute $f'(x)$. You must use the limit definition to receive credit.

$$f(x) = x^2 + 1$$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2+1-(x^2+1)}{h} = \lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{+1}\cancel{-x^2}\cancel{-1}}{h} = \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0}(2x+h) = \boxed{2x = f'(x)}$$

OR

$$f'(a) = \lim_{x\to a}\frac{x^2+1-(a^2+1)}{x-a} = \lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x+a)\cancel{(x-a)}}{\cancel{x-a}} = \lim_{x\to a} x+a = a+a = 2a$$

So $\boxed{f'(x) = 2x}$

24. Compute $\int_0^2 x^3\,dx$ using the Riemann sum definition. You must use the definition to receive any credit.

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = 0 + \frac{2}{n}i = \frac{2}{n}i$$

$$f(x_i) = \left(\frac{2}{n}i\right)^3 = \frac{8}{n^3}i^3$$

$$\int_0^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{8}{n^3}i^3\cdot\frac{2}{n} = \lim_{n\to\infty}\frac{16}{n^4}\sum_{i=1}^{n}i^3 = \lim_{n\to\infty}\frac{16}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2$$

$$= \lim_{n\to\infty}\frac{16(n^4+2n^3+n^2)}{4n^4} = \frac{16}{4} = \boxed{4}$$

25. Consider the function $f(x) = \dfrac{1}{\sqrt{x}}$ on the interval $[1,4]$.

a.) Verify that the Mean Value Theorem for Integrals applies to f.

b.) Find the value c guaranteed by the MVT.

a.) f is continuous on $[1,4]$ since it is a rational function and $[1,4]$ is in its domain.

b.) $f_{avg} = \frac{1}{4-1}\int_1^4 x^{-1/2}\,dx = \frac{1}{3}\cdot 2\sqrt{x}\Big|_1^4 = \frac{2}{3}(2-1) = \frac{2}{3}.$

$\frac{1}{\sqrt{x}} = \frac{2}{3} \Rightarrow \sqrt{x} = \frac{3}{2} \Rightarrow \boxed{x = \sqrt{\frac{3}{2}}}$

26. Find two positive numbers x and y satisfying $x + 4y = 100$, such that their product is a maximum.

$P = xy = y(100-4y) = -4y^2 + 100y$

$\frac{dP}{dy} = -8y + 100 = 0 \Rightarrow \boxed{\begin{array}{l} y = 12.5 \\ x = 100 - 4y = 50 \end{array}}$

27. Let f be continuous on the interval $[0,1]$. Prove that $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$.

$\left\{\begin{array}{l} u = 1-x \rightarrow x = 1-u \\ du = -dx \\ u(1) = 0 \\ u(0) = 1 \end{array}\right\}$

$\int_0^1 f(x)dx = \int_1^0 f(1-u)(-du) = -\int_1^0 f(1-u)\,du$

$= \int_0^1 f(1-u)\,du \quad \left\{\begin{array}{l} u = x \\ du = dx \end{array}\right\}$

$= \underline{\int_0^1 f(1-x)\,dx}$ ∎

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