M242 – Calculus I

Final Exam Review – Brief Solutions

1. Compute $\int_0^2 x^2 - x\,dx$ by using the Riemann sum definition of the integral.

$f(x) = x^2 - x$

$a = 0$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_i = 0 + i\frac{2}{n} = i\frac{2}{n}$

$f(x_i) = \left(i\frac{2}{n}\right)^2 - \left(i\frac{2}{n}\right) = i^2\frac{4}{n^2} - i\frac{2}{n}$

$$\int_0^2 x^2 - x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(i^2\frac{4}{n^2} - i\frac{2}{n}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{4}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty}\left(\frac{8}{n^3}\cdot\frac{(2n+1)(n+1)n}{6} - \frac{4}{n^2}\,\frac{n(n+1)}{2}\right)$$

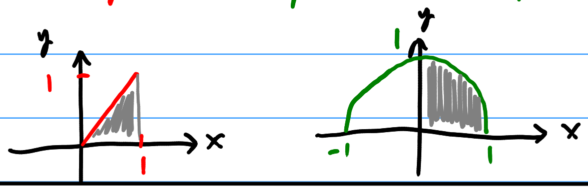
$$= 16/6 - 4/2 = \frac{8}{3} - 2 = \boxed{\frac{2}{3}}$$

2. Evaluate the integral by interpreting it in terms of areas.

$$\int_0^1 \left(x + \sqrt{1-x^2}\right)dx$$

$$\int_0^1 \left(x + \sqrt{1-x^2}\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{2}(1)(1) + \frac{1}{4}\pi(1^2) = \frac{1}{2} + \pi/4 = \boxed{\frac{2+\pi}{4}}$$

$y = x$ $\qquad$ $y = \sqrt{1-x^2} \rightarrow x^2 + y^2 = 1$



3. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

$$\int_4^6 f(x)dx = \int_4^0 f(x)dx + \int_0^6 f(x)\,dx = -\int_0^4 f(x)dx + \int_0^6 f(x)dx = -7 + 10 = 3.$$

4. Evaluate:

a.) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right)dx$ $= \sin(\pi/4)\cos(\pi/6) - \sin 0\cos 0 = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - 0\cdot 1 = \boxed{\frac{\sqrt{6}}{4}}$

b.) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx$ $= 0$

c.) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt$ $= -\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{3}\right)$

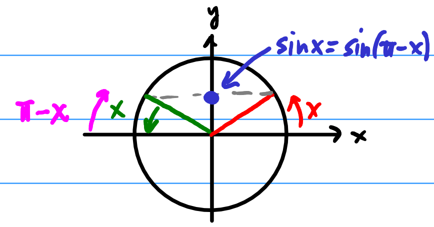
5. Compute $\displaystyle\int \frac{(\arctan x)^2}{1+x^2}\,dx.$

$\left\{\begin{array}{l} u = \arctan x \\ du = \frac{1}{1+x^2}dx \end{array}\right\}$ $= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\arctan x)^3 + C$

6. a.) If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x\, f(\sin x)\, dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx.$$

b.) Evaluate $\displaystyle\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\, dx.$



a.) $\left\{\begin{matrix} u = \pi - x, & u(\pi) = 0 \\ du = -dx, & u(0) = \pi \end{matrix}\right\}$

$$\int_0^\pi x f(\sin x)\,dx = -\int_\pi^0 (\pi - u)\, f(\sin(\pi - u))\,du$$
$$= \int_0^\pi \pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du$$
$$= \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx$$

$$\Rightarrow 2\int_0^\pi x f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx$$

$$\Rightarrow \int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx \quad \blacksquare$$

b.) $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$ by part a.).

$= \left\{\begin{matrix} u = \cos x & u(\pi) = -1 \\ du = -\sin x\,dx & u(0) = 1 \end{matrix}\right\} = \frac{\pi}{2}\displaystyle\int_{-1}^1 \frac{1}{1+u^2}\,du = \pi\int_0^1 \frac{1}{1+u^2}\,du$

$= \pi \arctan(u)\Big|_{u=0}^1 = \pi\left(\arctan(1) - \arctan(0)\right) = \pi\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi^2}{4}}$

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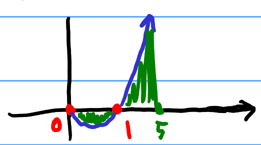
7. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find the displacement and the distance traveled by the particle during the time interval $[0, 5]$.

$v(t) = t^2 - t$



$s = \displaystyle\int_0^5 v(t)\,dt = \int_0^5 t^2 - t\,dt = \frac{1}{3}t^3 - \frac{1}{2}t^2\Big|_0^5 = \frac{125}{3} - \frac{25}{2}$

$= \dfrac{250 - 75}{6} = \boxed{\dfrac{175}{6}\text{ m}}$ displacement.

distance $= \displaystyle\int_0^1 -(t^2 - t)\,dt + \int_1^5 t^2 - t\,dt = -\frac{1}{3}t^3 + \frac{1}{2}t^2\Big|_0^1 + \frac{1}{3}t^3 - \frac{1}{2}t^2\Big|_1^5$

$= -\frac{1}{3} + \frac{1}{2} + \frac{125}{3} - \frac{25}{2} - \frac{1}{3} + \frac{1}{2} = \frac{175}{6} + 1 - \frac{2}{3} = \frac{175}{6} + \frac{1}{3} = \boxed{\frac{177}{6}\text{ m}}$

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8. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_0^8 r(t)\,dt$ represent?

$\int_0^8 r(t)\,dt$ = the total amount of oil consumed from Jan. 1, 2000 through Dec. 31, 2008.

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9. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate $\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta$.

$\left\{\begin{matrix} x = 2\sin\theta & u(\pi/2) = 2\sin\frac{\pi}{2} = 2 \\ dx = 2\cos\theta\,d\theta & u(0) = 2\sin(0) = 0 \end{matrix}\right\}$ so, $\displaystyle\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\cdot 6 = \boxed{3}$

10. If f is a continuous function such that

$$\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt$$

for all x, find and explicit formula for $f(x)$.

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt\right]$$

$$\rightarrow f(x) = x\cos x + \sin x + \frac{f(x)}{1+x^2}$$

$$\rightarrow \frac{(1+x^2)-1}{1+x^2} f(x) = x\cos x + \sin x \Rightarrow \boxed{f(x) = \frac{(1+x^2)(x\cos x + \sin x)}{x^2}}$$

11. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

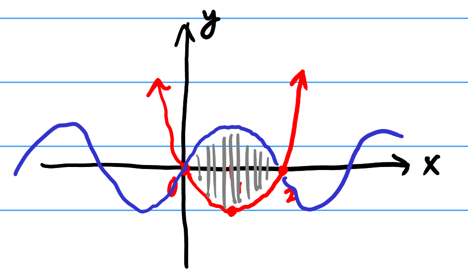
let $x = 1-u$, then $dx = -du$, $u(1)=0$, $u(0)=1$. Then $\int_0^1 f(x)dx = -\int_1^0 f(1-u)du = \int_0^1 f(1-u)du$

12. Find the area of the region bounded by the curves $y = \sin(\pi x/2)$ and $y = x^2 - 2x$.

$y = x^2-2x+1-1 = (x-1)^2-1$

$$A = \int_0^2 \sin\left(\frac{\pi x}{2}\right) - x^2 + 2x \, dx = -\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}x^3 + x^2 \Big|_0^2$$

$$= -\frac{2}{\pi}\cos\pi + \frac{2}{\pi}\cos 0 - \frac{8}{3} + 4 = \boxed{\frac{4}{\pi} + \frac{4}{3}}$$



13. Find the volume of the solid obtained by rotating the region bounded by the curves about the line $y = -1$.

$$y = x^2 + 1, \quad y = 9 - x^2$$

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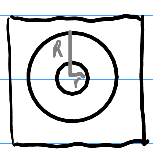


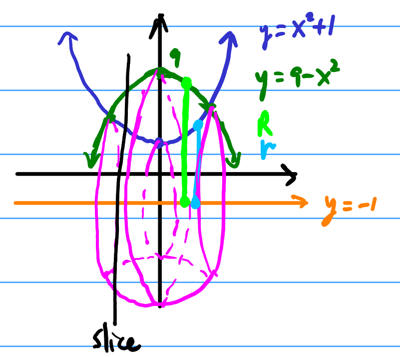
$$\text{Area} = \pi R^2 - \pi r^2 = \pi(9-x^2+1)^2 - \pi(x^2+1+1)^2$$

$$= \pi(100 - 20x^2 + x^4 - x^4 - 4x^2 - 4) = \pi(-24x^2 + 96)$$

Intersections: $x^2+1 = 9-x^2 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

$$\text{Volume} = \pi\int_{-2}^{2} -24x^2 + 96\,dx = 48\pi\int_0^2 -x^2+4\,dx = 48\pi\left(-\frac{1}{3}x^3 + 4x\right)\Big|_0^2 = 48\pi\left(8 - \frac{8}{3}\right)$$

$$= \frac{48\cdot 16}{3}\pi = 16^2\pi = \boxed{256\pi}$$

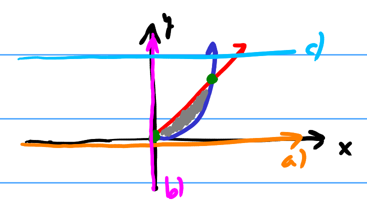


14. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines.

a.) the x-axis

b.) the y-axis

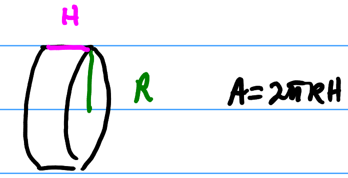
c.) $y = 2$



a.) $V = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

b.) $V = \int_0^1 \pi(\sqrt{y} - y)\,dy = \pi\left(\frac{2}{3} - \frac{1}{2}\right) = \pi/6$

c.) $V = \int_0^1 \pi\left((2-x^2)^2 - (2-x)^2\right)dx = \pi\int_0^1 4x - 3x^2 + x^4\,dx = \pi\left(2 - 1 + 1/5\right) = \frac{6\pi}{5}$



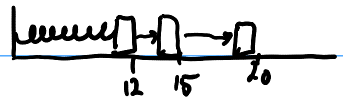
15. The following integral represents the volume of a solid of revolution. Describe the solid.

$$\int_0^4 2\pi \underbrace{(6-y)}_{R} \underbrace{(4y-y^2)}_{H}\, dy$$

Cylindrical Shells: $x = 4y - y^2$, rotated about the line $y=6$, from $y=0 \to y=4$.
also bounded by the y-axis.

16. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

$f(x) = kx$



$30 = k(3) \Rightarrow k = 10$

$W = \int_0^8 10x\,dx = 5x^2\Big|_0^8 = 5 \cdot 64 = 320$ J

17. Find the average value of the function $f(t) = \sec^2(t)$ on the interval $[0, \pi/4]$.

$f_{avg} = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} \sec^2 t\,dt = \frac{4}{\pi} \tan t\Big|_0^{\pi/4} = \frac{4}{\pi}(1-0) = 4/\pi$

18. If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x+h]$?

By MVT(I), there exists a number c with $x<c<x+h$ satisfying $f_{avg} = f(c)$.
Thus, taking the limit as $h \to 0$, $f_{avg} \to f(x)$.

19. Show that

$$\frac{d}{dx}\left(\frac{1}{2}\arctan(x) + \frac{1}{4}\ln\left(\frac{(x+1)^2}{x^2+1}\right)\right) = \frac{1}{(1+x)(1+x^2)}$$

$\frac{d}{dx}\left(\frac{1}{2}\arctan x + \frac{1}{4}\ln\left(\frac{(x+1)^2}{x^2+1}\right)\right) = \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{4}\frac{d}{dx}\left(2\ln(x+1) - \ln(x^2+1)\right) = \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{2}\frac{x}{x^2+1}$

$= \frac{1}{2}\left(\frac{(1-x)}{1+x^2} + \frac{1}{1+x}\right) = \frac{1}{2}\left(\frac{1-\cancel{x^2}+1+\cancel{x^2}}{(1+x^2)(1+x)}\right) = \frac{1}{2}\left(\frac{2}{(1+x^2)(1+x)}\right)$

$= \frac{1}{(1+x^2)(1+x)}$ ✓

20. At what point on the curve $y = [\ln(x+4)]^2$ is the tangent line horizontal?

$y' = \frac{2\ln(x+4)}{x+4} = 0$ when $x+4=1 \Rightarrow \underline{x=-3.}$ $\quad y(-3) = (\ln 1)^2 = 0.$

So the point is $(-3, 0)$.

21. If $f(x) = x + x^2 + e^x$, find $(f^{-1})'(1)$.

$f(0) = 1$ by inspection.

$(f^{-1})'(1) = \frac{1}{f'(0)}$ by the Inverse Function Theorem

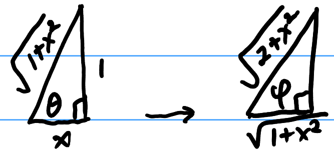
$f'(x) = 1 + 2x + e^x$

$f'(0) = 1 + 0 + 1 = 2$

So, $(f^{-1})'(1) = 1/2$.

22. Show that

$$\cos(\arctan(\sin(\underbrace{\operatorname{arccot} x}_{\theta}))) = \sqrt{\frac{x^2+1}{x^2+2}}.$$



$\sin\theta = \frac{1}{\sqrt{1+x^2}}$ $\quad\cos\varphi = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}}$ ✓