Name: Key
M242: Calculus I (Fall 2018)
Midterm Exam, part II

Read and follow all instructions. You may not use any electronic devices.

Part II (a): "Proofs"
Complete the following problems, showing enough work. Each problem is worth 10 points. Partial credit will be given when deserved.

1. You wish to prove that $\lim_{x\to 5}\big(10x-7\big)=43$. If you fix $\varepsilon>0$, what should you set δ equal to in order to finish the proof? Show enough work to justify your answer.

Suppose $0<|x-5|<\delta$.

Then, $|10x-7-43| = |10x-50| = 10|x-5| < 10\delta = 10\left(\frac{\varepsilon}{10}\right) = \varepsilon$

So, put $\boxed{\delta(\varepsilon)=\frac{\varepsilon}{10}}$.

2. Use the limit definition of the derivative to prove that $\frac{d}{dx}\left[x^5\right]=5x^4$. You must use the limit definition to receive any credit. Show enough work.

$f(x)=x^5$

$f'(a) = \lim_{x\to a}\frac{x^5-a^5}{x-a} = \lim_{x\to a}\left(x^4+ax^3+a^2x^2+a^3x+a^4\right) = a^4+a^4+a^4+a^4+a^4 = 5a^4$.

So, $\boxed{f'(x)=5x^4}$.

Part II (b): Applications

Complete the following problems, showing enough work. *Each problem is worth 10 points. Partial credit will be given when deserved.*

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3. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the interval. Then find the value(s) of c guaranteed by the theorem.

$$f(x) = x^3 - 6x^2 + 3x - 1, \quad 0 \le x \le 1$$

f is a polynomial, so f is continuous on $[0,1]$ and differentiable on $(0,1)$.

$f(1) = 1-6+3-1 = 4-7 = -3$

$f(0) = 0-0+0-1 = -1$

$m_s = \frac{f(1)-f(0)}{1-0} = \frac{-3-(-1)}{1-0} = \frac{-2}{1} = -2$

$f'(x) = 3x^2 - 12x + 3 = -2$

$3x^2 - 12x = -5$

$3(x^2 - 4x + 4) = -5 + 12$

$(x-2)^2 = \frac{7}{3}$

$x = 2 \pm \sqrt{\frac{7}{3}}$

only $x = 2 - \sqrt{\frac{7}{3}}$ is in $(0,1)$.

so $\boxed{c = 2 - \sqrt{\frac{7}{3}}}$

4. Consider the piecewise function

$$F(x) = \begin{cases} x^2 - 2x + 1 & x < 1 \\ k & x = 1 \\ \cos(x\pi) & x > 1 \end{cases}.$$

Is it possible for F to be continuous at $x = 1$? If so, what must k equal? Explain.

$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} x^2-2x+1 = 1-2+1 = 0$

$\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} \cos(x\pi) = \cos(\pi) = -1$

These are not equal, so F cannot be continuous at $x=1$.

5. Use a linear approximation or differentials to approximate the value of $\sqrt{3.99}$.

" You may leave your answer as a fraction. "

$f(x)=\sqrt{x}$ $\quad f'(x)=\dfrac{1}{2\sqrt{x}}$

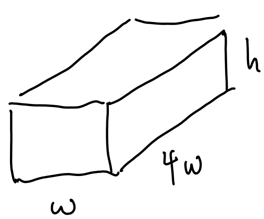
$f(4)=2$ $\quad f'(2)=\dfrac{1}{4}$

$L(x)=2+\frac{1}{4}(x-4)$

$L(3.99)=2+\frac{1}{4}(3.99-4)=2-\frac{1}{400}=\boxed{\dfrac{799}{400}}$

6. You wish to construct a rectangular box without a top such that the surface area of the box is 100 square inches. The base of the box must be a rectangle such that $\ell = 4w$. What are the dimensions of the box that maximize the volume? ($V = \ell \cdot w \cdot h$)

leave your answers as fractions.



$A=4w^2+2wh+8wh=4w^2+10wh=100$

So $h=\dfrac{100}{10w}-\dfrac{4w^2}{10w}=\dfrac{10}{w}-\dfrac{2}{5}w$

$V(w)=4w^2h=4w^2\left(\dfrac{10}{w}-\dfrac{2}{5}w\right)=40w-\dfrac{8}{5}w^3$

$\dfrac{dV}{dw}=40-\dfrac{24}{5}w^2=0 \Rightarrow w^2=\dfrac{200}{24}=\dfrac{25}{3}$

$\Rightarrow w=\dfrac{5}{\sqrt{3}}=\dfrac{5\sqrt{3}}{3}$ in

$\ell=\dfrac{20\sqrt{3}}{3}$ in

$h=\overset{2}{\cancel{10}}\cdot\dfrac{\sqrt{3}}{\cancel{5}}-\dfrac{2}{\cancel{5}}\cdot\dfrac{\cancel{5}\sqrt{3}}{3}$

$=\left(2-\dfrac{2}{3}\right)\sqrt{3}=\dfrac{4\sqrt{3}}{3}$ in

So, the dimensions are

$\dfrac{5\sqrt{3}}{3}\times\dfrac{20\sqrt{3}}{3}\times\dfrac{4\sqrt{3}}{3}$ in^3