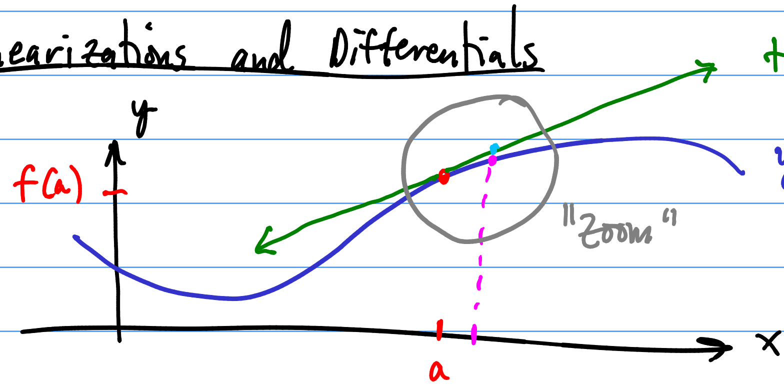
M242 9/14/18

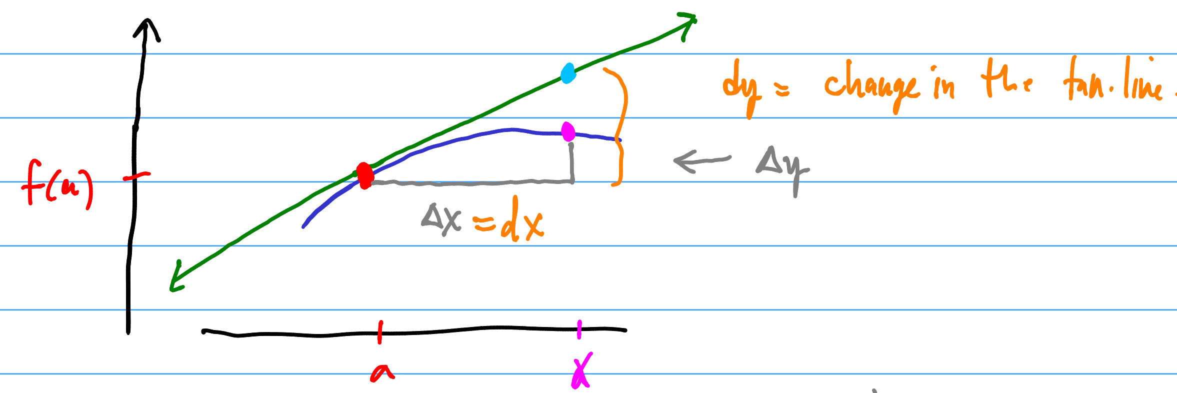
§2.9 – Linearizations and Differentials



We can use the tangent line to $y=f(x)$ at $x=a$ to approximate the values of f at points near a.

if x is near a,

$$f(x) \approx f(a) + f'(a)(x-a)$$



$x - a = dx$

$x = a + dx$

$f'(a) = \frac{dy}{dx}$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(a+dx) \approx f(a) + \frac{dy}{\cancel{dx}} \cancel{dx}$$

$$\boxed{f(a+dx) \approx f(a) + dy} \quad (*)$$

Def'n. The differential $dx = \Delta x = (x-a)$,

$$dy = f'(a)\,dx.$$

$$dy \neq \Delta y \quad \text{but} \quad dx = \Delta x.$$

Ex. $y = \sin x$ Find dy.

Find the linearization at $x = \pi/4$.

$$dy = \frac{dy}{dx}\,dx = \boxed{\cos x\,dx = dy}$$

$$y = f(a) + f'(a)(x-a)$$

$f(x) = \sin x$

$a = \pi/4$

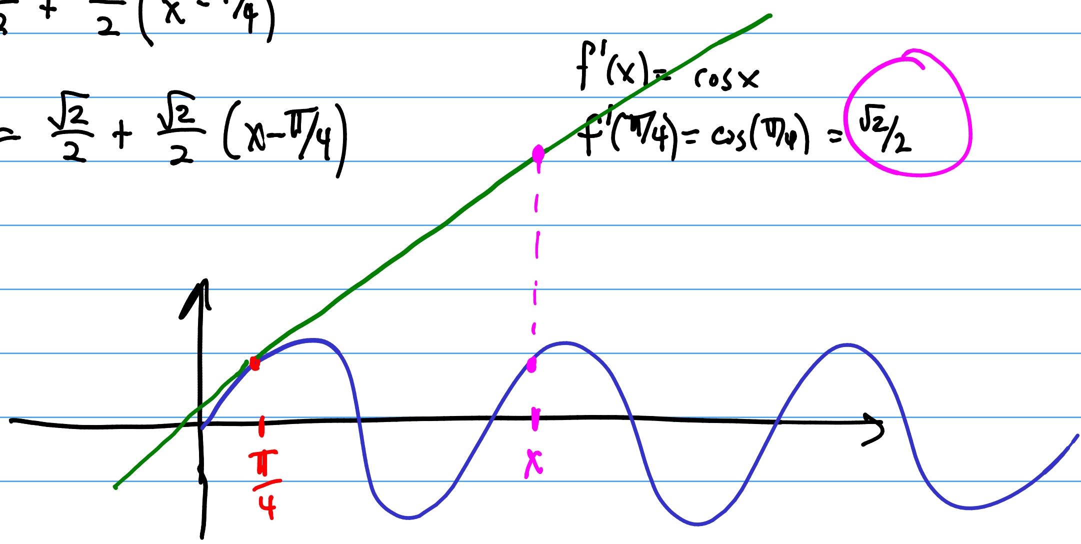
$f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$

$$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \pi/4\right)$$

$f'(x) = \cos x$

$f'(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$

$$L(x) = L_f(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \pi/4\right)$$



Ex. $(1.999)^5$ Approximate using linearization or differentials.

$f(x) = y = x^5 \rightsquigarrow f(2+dx) \approx f(2) + dy$

$a = 2$

$f(2) = 2^5 = 32$

$dx = -0.001$ $\quad f'(x) = 5x^4, \quad f'(a) = f'(2) = 5 \cdot 2^4 = 80$

$dy = f'(a)\,dx$,

$$f(1.999) \approx 32 + 80(-0.001) = 32 - 0.08$$

$$(1.999)^5 \approx 31.92$$

Ex. $\sqrt[3]{65}$ Approximate!

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad$ $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$

$a = 64 \qquad dx = 1$

$f(a) = 4$ $\qquad$ $f'(64) = \frac{1}{3(\sqrt[3]{64})^2} = \frac{1}{48}$

$$\sqrt[3]{65} \approx \sqrt[3]{64} + \frac{1}{48}\cdot 1 = 4 + \frac{1}{48} = \frac{193}{48} \approx 4.0208\bar{3}$$

Actual $\approx 4.02073\ldots$

Ex. Approximate $\sqrt{3.98}$

$$f(a+dx) \approx f(a) + dy$$

$f(x) = \sqrt{x}$ $\qquad$ $a = 4$ $\qquad$ $f(4) = \sqrt{4} = 2$ $\qquad$ $dx = -0.02$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad$ $f'(a) = \frac{1}{2\sqrt{4}} = 1/4$

$$\sqrt{3.98} \approx \sqrt{4} + \frac{1}{4}\left(-\frac{1}{50}\right) = 2 - \frac{1}{200} = \frac{399}{200} = 1.995$$

Ex. $V = \frac{4}{3}\pi r^3$

$dr = 0.1$ cm

What is the max. error $\underline{dV}$ when $r = 2$?

$$dV = \left(\frac{4}{3}\pi r^3\right)' dr = 4\pi r^2\, dr$$

$$dV\Big|_{r=2} = 4\pi(2)^2 \cdot 0.1 = 1.6\pi \text{ cm}^3 \leftarrow$$

$\left\{ \frac{dV}{V} = \text{rel. error.} \right.$