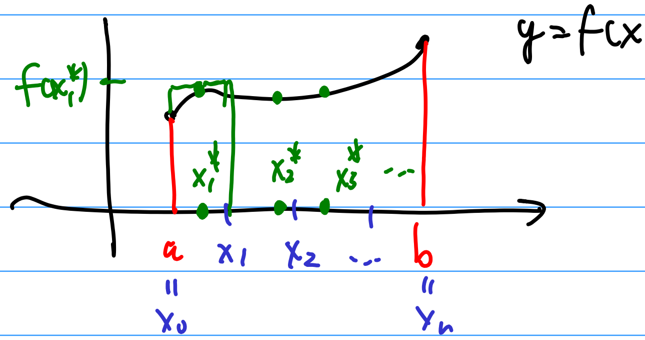
$\int_a^b f(x)\,dx$ = area under $y=f(x)$ on $[a,b]$.



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

Riemann Sum definition of definite integral.

Ex. $\int_1^2 x^2\,dx$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \boxed{\frac{1}{n}}$

$a = 1$

$x_i = a + i\Delta x = 1 + i\frac{1}{n}$

$f(x_i) = (1 + i\frac{1}{n})^2 = \boxed{1 + \frac{2}{n} i + \frac{1}{n^2} i^2}$

$$\int_1^2 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2}{n} i + \frac{1}{n^2} i^2\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} \underbrace{\sum_{i=1}^{n} 1}_{n} + \frac{2}{n^2} \underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} + \frac{1}{n^3} \underbrace{\sum_{i=1}^{n} i^2}_{\frac{n(n+1)(2n+1)}{6}} \right)$$

$1 + 2 + \dots + n$
$n + (n-1) + \dots + 1$
$(n+1) + (n+1) \dots$

$$= \lim_{n\to\infty} \left(\frac{n}{n} + \frac{\cancel{2}}{n^2} \cdot \frac{n(n+1)}{\cancel{2}} + \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \lim_{n\to\infty} \frac{n}{n} + \lim_{n\to\infty} \frac{n^2+n}{n^2} + \lim_{n\to\infty} \frac{2n^3+\cdots}{6n^3}$$

$$\frac{1}{1} + \frac{1}{1} + \frac{2}{6} = 2+\frac{1}{3} = \boxed{\frac{7}{3}}$$

We get $\int_1^2 x^2\,dx = \frac{7}{3}$

<u>Recall:</u> Let f be a continuous function on $[a,b]$.
An <u>antiderivative</u> of f is a function F on $[a,b]$ s.t.

$$F'(x) = f(x) \quad \text{for all } x \text{ in } [a,b].$$

<u>Recall:</u> <u>Mean Value Theorem</u>

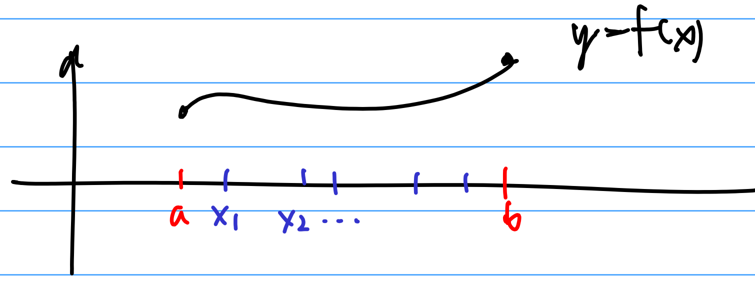
Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a number c such that $a<c<b$ and

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

$\int_a^b f(x)\,dx$ Suppose f is continuous on $[a,b]$.

Let F be any antiderivative of f.

Let $a=x_0<x_1<x_2<\cdots<x_n=b$ be a partition of $[a,b]$.

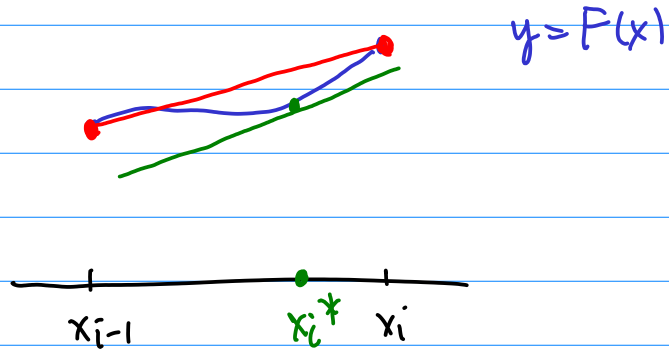


Consider $\underbrace{\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]}$ $= \cancel{F(x_1)} - F(x_0) + \cancel{F(x_2)} - \cancel{F(x_1)} + \cancel{F(x_3)} - \cancel{F(x_2)}$
$+ \cdots + F(x_n) - \cancel{F(x_{n-1})}$

$$= F(x_n) - F(x_0) = \underbrace{F(b) - F(a)}$$

Telescoping sum!

Zoom in on one interval



$\underline{\text{MVT}}$: There is a x_i^* in (x_{i-1}, x_i) such that

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{\boxed{x_i - x_{i-1}}\ \Delta x_i}$$

So, $$F(x_i) - F(x_{i-1}) = \underbrace{F'(x_i^*)}_{f(x_i^*)} \Delta x_i$$

Putting it all together,

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n}\left[F(x_i) - F(x_{i-1})\right] = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$

$$= \int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus

Let f be a continuous function on $[a,b]$, and let F be any antiderivative of f. Then,

$$\text{I.} \quad \frac{d}{dx}\int_a^x f(u)\,du = f(x)$$

$$\text{II.} \quad \int_a^b f(x)\,dx = F(b) - F(a)$$

OR

$$\int_a^b \frac{d}{dx}\left[F(x)\right] dx = F(b) - F(a)$$

Proof. Part II was done above.

Part I.

$$\cancel{g(x)} = \int_a^x f(u)\,du \overset{\text{FTC II}}{=} F(x) - F(a)$$

Then, $$\frac{d}{dx}\left[\int_a^x f(u)\,du\right] = \frac{d}{dx}\left[F(x) - F(a)\right]$$

$$= F'(x) - 0$$
$$= f(x). \quad \blacksquare$$

Ex. $$\int_1^2 x^2\,dx = \frac{1}{3}x^3\Big|_{x=1}^{2} = \frac{1}{3}2^3 - \frac{1}{3}1^3 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Ex. $$\int_{\pi/4}^{\pi/2} \sin(x)\,dx = -\cos(x)\Big|_{\pi/4}^{\pi/2} = -\cos(\pi/2) + \cos(\pi/4) = 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

Ex. $\frac{d}{dx}\int_{\pi/4}^{x} \sin(t)\,dt = \sin(x)$

$$\frac{d}{dx}\left[-\cos t\Big|_{\pi/4}^{x}\right] = \frac{d}{dx}\left[-\cos x + \frac{\sqrt{2}}{2}\right] = \sin x$$

Ex. $\frac{d}{dx}\int_{0}^{x} \sin(t^2)\,dt = \sin(x^2)$ by FTC part I

Ex. $\frac{d}{dx}\int_{0}^{x^2} \sin(t)\,dt = \frac{d}{dx}\left[-\cos(t)\Big|_{0}^{x^2}\right] = \frac{d}{dx}\left[-\cos(x^2) + 1\right]$

$$= 2x\sin(x^2)$$

So, $\frac{d}{dx}\int_{a}^{u(x)} f(t)\,dt = u'(x)\,f(u)$