Name: Key

M243: Calculus II (Fall 2018)
Instructor: Justin Ryan
Final Exam

WICHITA STATE UNIVERSITY

Read and follow all instructions. You may not use any electronic devices. You may use a single two-sided 8.5 by 11 inch page of your own hand-written notes.

1–3. Computations [5 points each]
Compute the following integrals. Show enough work. Be sure to treat any improper integrals properly.

1. $\int \frac{1}{x^2+x-6}\,dx = \int \frac{1}{(x+3)(x-2)}\,dx = \int \frac{A}{x+3} + \frac{B}{x-2}\,dx = A\ln|x+3| + B\ln|x-2| + C$

PFD: $1 = A(x-2) + B(x+3)$

$x=2:\ 1 = 5B \Rightarrow B = 1/5$

$x=-3:\ 1 = -5A \Rightarrow A = -1/5$

$\Rightarrow \quad = \boxed{-\frac{1}{5}\ln|x+3| + \frac{1}{5}\ln|x-2| + C}$

OR

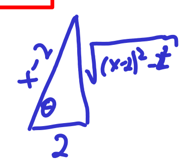
$x^2+x-6 = (x+1/2)^2 - 6 - 1/4 = (x+1/2)^2 - (5/2)^2 \Rightarrow \int \frac{-1}{(5/2)^2-(x+1/2)^2}\,dx = \frac{2}{5}\operatorname{arctanh}\left(\frac{x+1/2}{5/2}\right) + C$

$= \boxed{\frac{2}{5}\operatorname{arctanh}\left(\frac{2x+1}{5}\right) + C}$

2. $\int \frac{1}{\sqrt{x^2-4x}}\,dx = \int \frac{1}{\sqrt{(x^2-4x+4)-4}}\,dx = \int \frac{1}{\sqrt{(x-2)^2-2^2}}\,dx$

let: $\left(\frac{x-2}{2}\right) = \sec\theta$

$\frac{1}{2}dx = \sec\theta\tan\theta$



$= \int \frac{2\sec\theta\tan\theta}{\sqrt{(2\sec\theta)^2-2^2}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \boxed{\ln\left|\frac{x-2}{2} + \frac{\sqrt{x^2-4x}}{2}\right| + C}$

3. $\int_0^5 \frac{x-2+2}{x-2}\,dx = \int_0^5 1 + \frac{2}{x-2}\,dx = 5 + 2\int_0^5 \frac{1}{x-2}\,dx = 5 + 2\int_0^2 \frac{1}{x-2}\,dx + 2\int_2^5 \frac{1}{x-2}\,dx$

but these integrals diverge by the p-test.

Written Problems [10 points each]
Complete each problem, showing enough work. Partial credit will be given when deserved.

4. Find the arc length of the curve $y = 2\ln\left(\sin(\frac{1}{2}x)\right)$ on the interval $[\frac{\pi}{3}, \pi]$.

$$\frac{dy}{dx} = \cancel{2} \cdot \frac{\frac{1}{\cancel{2}}\cos(\frac{1}{2}x)}{\sin(\frac{1}{2}x)} = \cot(1/2\, x)$$

$$\left(\frac{dy}{dx}\right)^2 = \cot^2(\tfrac{1}{2}x)$$

$$\sqrt{1+\cot^2(1/2\, x)} = \sqrt{\csc^2(1/2\, x)} = |\csc(1/2\, x)| > 0 \text{ on } [\pi/3, \pi].$$

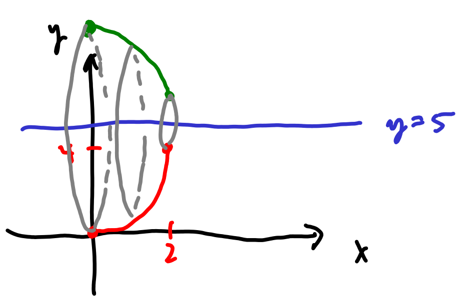
So $$s = \int_{\pi/3}^{\pi} \csc(1/2\, x)\, dx = 2\int_{\pi/6}^{\pi/2} \csc u\, du = -2\ln|\csc u + \cot u| \Big|_{\pi/6}^{\pi/2} = -2\ln|1+0| + 2\ln|2+\sqrt{3}|$$

$u = \frac{1}{2}x \quad u(\pi) = \pi/2$

$du = \frac{1}{2}dx \quad u(\pi/3) = \pi/6$

$$= \boxed{2\ln(2+\sqrt{3})}$$

5. Consider the surface obtained by rotating the curve $y = x^2$, $0 \le x \le 2$ about the line $y = 5$. Write an integral that represents the surface area of the surface. **Do NOT evaluate the integral.**



$$A = \pi\int_{x=0}^{x=2} 2(5-y)\, ds = \boxed{\pi\int_0^2 2(5-x^2)\sqrt{1+4x^2}\, dx}$$

6–7. Consider the polar curve given by $r = \cos(4\theta)$.

6. Find the area enclosed by one loop of the curve.

$\cos(4\theta) = 0 \Rightarrow 4\theta = \pm \pi/2 \Rightarrow \theta = \pm \pi/8$

$$A = \int_{-\pi/8}^{\pi/8} \frac{1}{2} r^2 \, d\theta = 2\int_0^{\pi/8} \frac{1}{2}\cos^2(4\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/8} 1 + \cos(8\theta)\, d\theta = \frac{1}{2}\left(\theta - \frac{1}{8}\sin(8\theta)\right)\Big|_0^{\pi/8}$$

$$= \frac{1}{2}\left(\pi/8 - \frac{1}{8}(\sin\pi - \sin 0)\right) = \boxed{\frac{\pi}{16}}$$

7. Write down an integral that represents the arc length of one loop of the curve. **Do NOT evaluate the integral.**

$\dot{r} = -4\sin(4\theta)$

$\dot{r}^2 = 16\sin^2(4\theta) \qquad r^2 = \cos^2(4\theta)$

$$\boxed{s = 2\int_0^{\pi/8} \sqrt{16\sin^2(4\theta) + \cos^2(4\theta)}\, d\theta}$$

8. Solve the initial value problem.

$$\begin{cases} \dfrac{dy}{dt} - 2ty = t^3, \\ y(0) = 1 \end{cases}$$

Linear! $p = -2t$ $\quad \mu = e^{\int -2t\,dt} = e^{-t^2}$

$$y = e^{t^2} \int t^3 e^{-t^2}\, dt + Ce^{t^2}$$

$$\left\{ \begin{array}{l} u = -t^2 \\ du = -2t\,dt \end{array} \quad t^3\,dt = -t^2(-t\ dt) = \tfrac{1}{2} u\,du \right\}$$

$$= e^{t^2} \underbrace{\int \tfrac{1}{2} u e^u\, du}_{\text{IbP!}} + Ce^{t^2} = \tfrac{1}{2} e^{t^2}(ue^u - e^u) + Ce^{t^2} = \tfrac{1}{2} e^{t^2}\left(-t^2 e^{-t^2} - e^{-t^2}\right) + Ce^{t^2}$$

$$y = -\tfrac{1}{2} t^2 - \tfrac{1}{2} + Ce^{t^2}$$

$y(0) = C - \tfrac{1}{2}$ so $C = \tfrac{3}{2}$ and $\boxed{y = -\tfrac{1}{2} t^2 - \tfrac{1}{2} + \tfrac{3}{2} e^{t^2}}$

9. Solve the initial value problem.

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2 - 2x}{4y} \\ y(2) = -\dfrac{3}{2} \end{cases}$$

$$\int 2y\, dy = \int \left(\tfrac{3}{2} x^2 - x\right) dx$$

$$y^2 = \tfrac{1}{2} x^3 - \tfrac{1}{2} x^2 + C$$

$$\left(-\tfrac{3}{2}\right)^2 = \tfrac{1}{2}(2)^3 - \tfrac{1}{2}(2^2) + C \quad \Rightarrow \quad C = \tfrac{9}{4} - 4 + 2 = \tfrac{1}{4}$$

So $\boxed{y = -\sqrt{\tfrac{1}{2} x^3 - \tfrac{1}{2} x^2 + \tfrac{1}{4}}}$

10-11. Consider the function $f(x) = \sin(x^2)$.

10. Find the MacLaurin series for f using your favorite method.

$$\sin(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u^{2n+1} \Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}}$$

11. Find a series representation of $\int_0^1 f(x)\,dx$. How many terms are needed to approximate the integral correct to 4 decimal places?

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}\,dx = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} x^{4n+3}\right)\Bigg|_0^1$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}} = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{\underbrace{11\cdot 5!}} - \frac{1}{\underbrace{15\cdot 7!}} + - \cdots$$

$11\cdot 120 = 1320$ $\quad 15\cdot 5040 = 75600 > 10,000 = 10^4$

$$\begin{array}{r} 1 \\ 720 \\ 7 \\ \hline 5040 \\ 15 \\ \hline 25200 \\ 5040 \\ \hline 75600 \end{array}$$

So only the first $\underline{\underline{3}}$ terms are needed to approximate $\int_0^1 \sin(x^2)dx$ within 4 decimals. :)

12. Find the Taylor series for $y = \ln x$ centered at $x_0 = 1$ using your favorite method.

$$y' = \frac{1}{x} = \frac{1}{x+1-1} = \frac{1}{1+(x-1)} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\ln x = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$$ but $\ln(1) = 0$ so $C = 0$,

and $$\boxed{\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n}$$

13. Find the radius and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+2)!}$$

RaT: $\left| \frac{2^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{2^n} \right| = \frac{2}{n+3} \xrightarrow{n\to\infty} 0$ so $\boxed{R = \infty \text{ and } I = (-\infty, \infty).}$

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