M243: Calculus II

Final Exam Review Guide - Brief Solutions

1. Evaluate $\int_0^1 e^x\,dx$ by using the Riemann sum definition and noticing that it's a geometric sum. You may check your answer using the Fundamental Theorem, but to receive credit you must do the Riemann sum.

Def'n: $\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(e^{1/n}\right)^k \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} e^{1/n} \underbrace{\sum_{k=1}^{n} \left(e^{1/n}\right)^{k-1}}_{\text{Geometric!}}$

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$x_k = a + k\Delta x = 0 + k\frac{1}{n} = \frac{k}{n}$

$f(x_k) = e^{x_k} = e^{k/n} = \left(e^{1/n}\right)^k$

$= \lim_{n\to\infty} \frac{1}{n} e^{1/n} \frac{1-\left(e^{1/n}\right)^n}{1-e^{1/n}} = \lim_{n\to\infty} \frac{1/n\left(e^{1/n}(1-e)\right)}{1-e^{1/n}} = \frac{0}{0}$ L'H

$= (1-e) \lim_{n\to\infty} \frac{(1/n)' e^{1/n} + \frac{1}{n} e^{1/n} \cancel{(1/n)'}}{-(1/n)' \cancel{e^{1/n}}} = (1-e) \lim_{n\to\infty} -(1+1/n)$

$= -(1-e) = e-1$ ✓

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2. Find the MacLaurin series for $f(x) = e^{-x^2}$ and use it to evaluate $\int_0^1 e^{-x^2}\,dx$ correct to 4 decimal places.

$e^u = \sum_{n=0}^{\infty} \frac{1}{n!} u^n$ w/ RoC $= \infty$, so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ is the MS.

$\int_0^1 e^{-x^2}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!} x^{2n+1} \Big|_0^1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!} - 0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!}$

This is an alternating series, so the partial sums are bounded by the abs. value of the next term.

We want $\left|\frac{(-1)^N}{(2N+1)\,N!}\right| < \frac{1}{1000} \Rightarrow (2N+1)N! \geq 1000$

when $N=5$, $(2\cdot5+1)5! = 11\cdot120 = 1320 > 1000$. So the maximum error in the partial sum S_4 will be $\frac{1}{1320}$.

Thus,

$\int_0^1 e^{-x^2}dx \approx \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)n!} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{7\cdot6} + \frac{1}{9\cdot24} = \frac{5651}{7560} \approx 0.7475$

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3. Find the Taylor series for $y = \ln x$ centered at $x_0 = 1$. What is its interval of convergence?

n	$f^{(n)}(x)$	$\xrightarrow{x=1}$	$f^{(n)}(1)$
0	$\ln x$		0
1	$1/x$		$1 = 0!$
2	$-1/x^2$		$-1 = -1!$
3	$+2/x^3$		$2 = 2!$
4	$-6/x^4$		$-6 = -3!$
$\vdots$	$\vdots$		$\vdots$
n	$\dfrac{(-1)^{n+1}(n-1)!}{x^n}$		$(-1)^{n-1}(n-1)!$

so, $\ln x = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$

<u>RaT</u>: $\left| \dfrac{\cancel{(-1)^n}(x-1)^{\cancel{n+1}}}{n+1} \cdot \dfrac{n}{\cancel{(-1)^{n-1}}\,\cancel{(x-1)^n}} \right| = \left(\dfrac{n}{n+1}\right)|x-1| \xrightarrow{n\to\infty} |x-1|$

so this series converges when $|x-1| < 1$

The interval is $-1 < x-1 < 1 \rightarrow 0 \underset{?}{\le} x \underset{?}{\le} 2$

Now check the endpoints: when $(x-1) = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$

This <u>diverges</u> by the p-test.

when $(x-1) = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ <u>converges</u> by AST.

So the interval of convergence is $\boxed{0 < x \le 2}$

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4. Find the Taylor series for $y = 3x^3 - 2x^2 + 9x + 4$ centered at $x_0 = 2$.

$f(x) = 3x^3 - 2x^2 + 9x + 4 \qquad f(2) = 24 - 8 + 18 + 4 = 38$

$f'(x) = 9x^2 - 4x + 9 \qquad f'(2) = 36 - 8 + 9 = 37$

$f''(x) = 18x - 4 \qquad f''(2) = 36 - 4 = 32$

$f'''(x) = 18 \qquad f'''(2) = 18$

$f^{(n)}(x) = 0$ for $n > 3$.

So $y = 3x^3 - 2x^2 + 9x + 4 = 38 + 37(x-2) + \frac{32}{2!}(x-2)^2 + \frac{18}{3!}(x-2)^3$

or $\boxed{y = 38 + 37(x-2) + 16(x-2)^2 + 3(x-2)^3}$

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5. Use series to evaluate the limit, $\displaystyle\lim_{x\to 0} \frac{x^3 - 3x + 3\arctan(x)}{x^5}$.

First, find the Maclaurin series for arctangent: $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \displaystyle\sum_{n=0}^{\infty}(-x^2)^n$

so $\arctan(x) = C + \displaystyle\sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

But $C = 0$ since $\arctan(0) = 0$. so $\arctan(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - + \dots$

Then,

$\displaystyle\lim_{x\to 0} \frac{x^3 - 3x + 3\arctan x}{x^5} = \lim_{x\to 0} \frac{\cancel{x^3} - \cancel{3x} + \cancel{3x} - \cancel{x^3} + \frac{3}{5}x^5 - \frac{3}{7}x^7 + \dots}{x^5} = \lim_{x\to 0}\left(\frac{3}{5} - \frac{3}{7}x^2 + - \dots\right) = \boxed{\frac{3}{5}}$

6. Use series to evaluate the limit, $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

Similar to the last problem. We need the first few terms of the Maclaurin series for tan(x).

$$\left.\begin{array}{ll} f(x) = \tan x & f(0) = 0 \\ f'(x) = \sec^2 x & f'(0) = 1 \\ f''(x) = 2\sec^2 x \tan x & f''(0) = 2\cdot 1\cdot 0 = 0 \\ f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x & f'''(0) = 4\cdot 1\cdot 0 + 2\cdot 1 = 2 \end{array}\right\} \text{ So, } \tan x = x + \tfrac{1}{3}x^3 + \cdots$$

Then $\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{(\cancel{x} + \frac{1}{3}x^3 + \cdots) - \cancel{x}}{x^3} = \lim_{x\to 0}\left(\frac{1}{3} + \cdots\right) = \boxed{\frac{1}{3}}$

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7. Find the radius of convergence of the series, $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$ ← very important omission!

RaT: $\left| \frac{(2n+2)!}{(n+1)!^2} \cdot \frac{n!^2}{(2n)!} \right| = \frac{(2n+2)(2n+1)\cancel{(2n)!}\cdot \cancel{(n!)^2}}{(n+1)^2 \cancel{(n!)^2} \cdot \cancel{(2n)!}} = \frac{4n^2+6n+2}{n^2+2n+1} \xrightarrow{n\to\infty} 4$

So the radius of convergence is $R = 1/4$

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8. Find the MacLaurin series for $y = \sqrt{1+x^4}$ and use it to approximate $\int_0^1 \sqrt{1+x^4}\, dx$.

Find the M.S. for $\sqrt{1+u}$ first, then swap.

$$\left.\begin{array}{ll} f(u) = \sqrt{1+u} & f(0) = 1 \\ f'(u) = \frac{1}{2\sqrt{1+u}} & f'(0) = \frac{1}{2} \\ f''(u) = \frac{-1}{4\sqrt{1+u}^3} & f''(0) = \frac{-1}{4} \\ f'''(u) = \frac{3}{8\sqrt{1+u}^5} & f'''(0) = \frac{3}{8} \\ f^{(4)}(u) = \frac{-15}{16\sqrt{1+u}^7} & f^{(4)}(0) = \frac{-15}{16} \end{array}\right\}$$

Thus $\sqrt{1+u} = 1 + \cancel{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)(n-1)\cdots 1}{2^n\, n!}}\, u^n$

meanwhile, $\cancel{(n+1)(n-1)(n-3)\cdots 1 = \frac{(n+1)!}{n\cdot(n-2)\cdot(n-4)\cdots 2}}$

$\vdots$

$f^{(n)}(u) = \cancel{\frac{(-1)^{n+1}(n+1)(n-1)\cdots(1)}{2^n \sqrt{1+u}^{\,2n-1}}} \rightarrow f^{(n)}(0) = \cancel{\frac{(-1)^{n+1}(n+1)\cdot(n-1)\cdots 1}{2^n}}$

Maybe not... Regardless, we can still use the first few terms.

So the M.S. for $\sqrt{1+x^4} = 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{5\cdot 3\cdot 1}{16\cdot 4\cdot 3\cdot 2}x^{16} + - \cdots$

$= 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{1}{128}x^{16} + - \cdots$ and

$\int_0^1 \sqrt{1+x^4}\,dx \approx x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \frac{1}{2176}x^{17} + - \cdots \Big|_0^1 = \frac{1388111}{1272960} \approx \boxed{1.0905}$

10. Consider the parametric curve $\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle$. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\left.\begin{aligned}\dot{y} &= (t\sin t)' = \sin t + t\cos t\\ \dot{x} &= (t\cos t)' = \cos t - t\sin t\end{aligned}\right\}\quad \boxed{\frac{dy}{dx} = \frac{\sin t + t\cos t}{\cos t - t\sin t}}$$

$$\left(\frac{dy}{dx}\right)^{\bullet} = \left(\frac{\sin t + t\cos t}{\cos t - t\sin t}\right)' = \frac{(\cos t - t\sin t)(\cos t + \cos t - t\sin t) - (\sin t + t\cos t)(-\sin t - \sin t - t\cos t)}{(\cos t - t\sin t)^2}$$

$$= \frac{\left(2\cos^2 t - \cancel{t\sin t\cos t} - \cancel{2t\sin t\cos t} + t^2\sin^2 t\right) - \left(-2\sin^2 t - \cancel{t\cos t\sin t} - \cancel{2t\cos t\sin t} - t^2\cos^2 t\right)}{(\cos t - t\sin t)^2}$$

$$= \frac{2 + t^2}{(\cos t - t\sin t)^2}$$

so, $\boxed{\frac{d^2y}{dx^2} = \frac{(dy/dx)^{\bullet}}{\dot{x}} = \frac{2+t^2}{(\cos t - t\sin t)^3}}$

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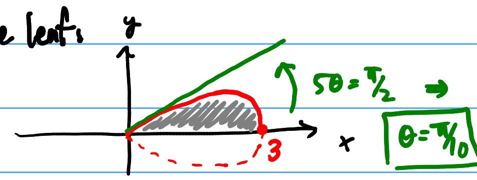
11. Find the area enclosed by the curve $r^2 = 9\cos(5\theta)$.

$A = \int_a^b \frac{1}{2} r^2\, d\theta$ Need to determine a and b.

one leaf:



$$= 2\cdot\frac{1}{2}\int_0^{\pi/10} r^2\, d\theta = \int_0^{\pi/10} 9\cos(5\theta)\, d\theta = \frac{9}{5}\sin(5\theta)\Big|_0^{\pi/10} = \frac{9}{5}\sin\frac{\pi}{2} - \frac{9}{5}\sin 0 = \boxed{\frac{9}{5}}$$ per leaf.

There will be 10 leaves, so the total area enclosed is $\frac{90}{5}$.
(The question will be stated more clearly on the exam.)

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12. Find the length of the curves.

a.) $\mathbf{r}(t) = \langle 3t^2, 2t^3\rangle, \quad 0 \le t \le 1$

b.) $r = \sin^3(\theta/3), \quad 0 \le \theta \le \pi$

$$s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

a.)
$$\left.\begin{aligned} x &= 3t^2 & \dot{x} &= 6t & \dot{x}^2 &= 36t^2\\ y &= 2t^3 & \dot{y} &= 6t^2 & \dot{y}^2 &= 36t^4\end{aligned}\right\}\ \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{36t^2(1+t^2)} = 6t\sqrt{1+t^2}$$

$$s = \int_0^1 6t\sqrt{1+t^2}\, dt \quad \left\{\begin{aligned} u &= 1+t^2 & u(1) &= 2\\ du &= 2t\,dt & u(0) &= 1\end{aligned}\right\} = 3\int_1^2 \sqrt{u}\, du = \cancel{3}\cdot\frac{2}{\cancel{3}}u^{3/2}\Big|_1^2$$

$$= \boxed{2\sqrt{8} - 2}$$

b.) $r = \sin^3\left(\frac{\theta}{3}\right), \quad 0 \le \theta \le \pi$

Recall, polar parametrization is $\left.\begin{aligned} x &= r\cos\theta = r(\theta)\cos\theta \\ y &= r\sin\theta = r(\theta)\sin\theta \end{aligned}\right\}$ now parametrized by θ!

$\dot{x} = \dot{r}\cos\theta - r\sin\theta \qquad \dot{x}^2 = (\dot{r})^2\cos^2\theta - 2\dot{r}r\cos\theta\sin\theta + r^2\sin^2\theta$

$\dot{y} = \dot{r}\sin\theta + r\cos\theta \qquad + \ \dot{y}^2 = (\dot{r})^2\sin^2\theta + 2\dot{r}r\cos\theta\sin\theta + r^2\cos^2\theta$

$\dot{x}^2 + \dot{y}^2 = (\dot{r})^2 + r^2$

so, $s = \int_0^\pi \sqrt{(\dot{r})^2 + r^2}\, d\theta = \int_0^\pi \sqrt{\left(\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right)^2 + \left(\sin^2\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right)\right)^2}\, d\theta$

$= \int_0^\pi \left|\sin\left(\frac{\theta}{3}\right)\right| d\theta = \int_0^\pi \sin\left(\frac{\theta}{3}\right) d\theta = -3\cos\left(\frac{\theta}{3}\right)\Big|_0^\pi$

$= -3\cos\left(\pi/3\right) + 3\cos 0 = 3 - \frac{3}{2} = \boxed{\frac{3}{2}}$

13. Find an equation of the curve that satisfies $y' = 4x^3y$ and whose y-intercept is 7.

$\frac{dy}{dx} = 4x^3y \rightarrow \int \frac{1}{y}\, dy = \int 4x^3\, dx \rightarrow \ln y = x^4 + C$

$\rightarrow y = Ce^{x^4}$

$y(0) = C = 7$

so $\boxed{y = 7e^{x^4}}$

14. Find the solution of the initial value problem.

$$\begin{cases} \dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \\ u(0) = -5 \end{cases}$$

$\int 2u\, du = \int (2t + \sec^2 t)\, dt$

$u^2 = t^2 + \tan t + C$

$u(0) = -5 \rightarrow 25 = C$, and $\boxed{u(t) = -\sqrt{t^2 + \tan t + 25}}$

15. Find the general solution $u = u(t)$ of the differential equation, $\dfrac{du}{dt} = 2 + 2u + t + tu$.

$\frac{du}{dt} - (2+t)u = 2+t$ Linear! $\mu = e^{\int p\, dt} = e^{\int -2-t\, dt} = e^{-2t - \frac{1}{2}t^2} = e^{-\left(\frac{1}{2}t^2 + 2t\right)}$

$u = \frac{1}{\mu}\int \mu g\, dt + \frac{C}{\mu} = e^{\frac{1}{2}t^2 + 2t}\int \underbrace{e^{-\left(\frac{1}{2}t^2 + 2t\right)}}_{u}\underbrace{(2+t)\, dt}_{-du} + Ce^{\frac{1}{2}t^2 + 2t} = \boxed{-1 + Ce^{\frac{1}{2}t^2 + 2t} = u}$