Name: Key

M243: Calculus II (Fall 2018)

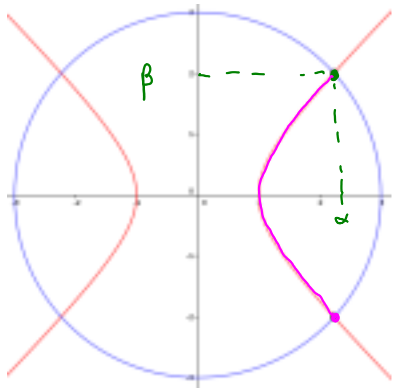
Midterm Exam, part II

Read and follow all instructions. You may not use any electronic devices, but you may use one 3×5 in^2 index card of your own hand-written notes.

Part I: Applications

In each of the following problems, set up and simplify an integral that represents the desired quantity. Do **NOT** *evaluate the integrals.*

1. Write an integral that represents the portion of the arc length of the hyperbola $x^2 - y^2 = 1$, $x > 0$, that is cut off by the circle $x^2 + y^2 = 9$.



Solve for (α, β):

circle: $x^2 = 9 - y^2$

hyperbola: $x^2 = 1 + y^2$

$$9 - y^2 = 1 + y^2$$
$$8 = 2y^2$$
$$y^2 = 4$$
$$y = \pm 2 \quad : \quad \boxed{\beta = 2}$$

Then $x^2 = 1 + 2^2 = 5$

so $x = \boxed{\sqrt{5} = \alpha}$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{d}{dx}\left[x^2 - y^2 = 1\right] \rightarrow 2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}, \text{ so } \frac{dx}{dy} = \frac{y}{x}$$

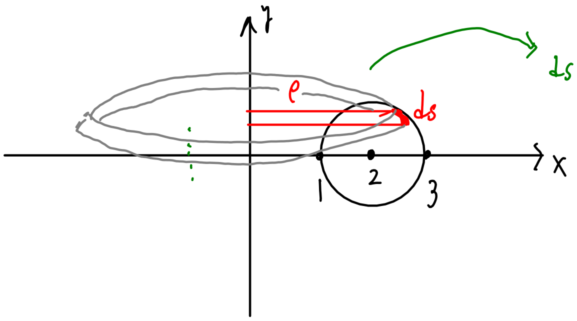
solving for y, $y = \pm\sqrt{x^2 - 1}$

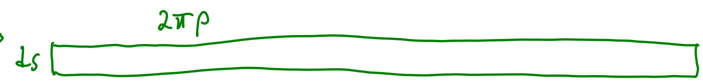
solving for x, $x = \sqrt{y^2 + 1}$

$$s = \int ds = \int_{-2}^{2} \sqrt{1 + \frac{y^2}{y^2+1}}\, dy = 2\int_{1}^{\sqrt{5}} \sqrt{1 + \frac{x^2}{x^2-1}}\, dx$$

$$\boxed{s = \int_{-2}^{2} \sqrt{\frac{2y^2+1}{y^2+1}}\, dy} \quad \text{or} \quad \boxed{s = 2\int_{1}^{\sqrt{2}} \sqrt{\frac{2x^2-1}{x^2-1}}\, dx}$$

2. Write an integral that represents the surface area of a torus with major radius $R = 2$ and minor radius $r = 1$.
[Recall: such a torus is obtained by rotating the circle $(x - R)^2 + y^2 = r^2$ about the y-axis.]





$$\rho = x$$

$$ds = \sqrt{1 + (dy/dx)^2}\, dx$$

$$y = \sqrt{r^2 - (x-R)^2}$$

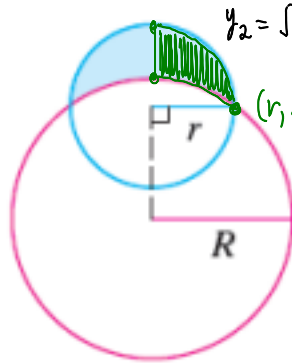
$$\frac{dy}{dx} = \frac{-(x-R)}{\sqrt{r^2-(x-R)^2}}$$

$$ds = \sqrt{1+(dy/dx)^2}\, dx = \frac{r}{\sqrt{r^2-(x-R)^2}}\, dx$$

So, $$A = 2\int_1^3 \frac{xr}{\sqrt{r^2-(x-R)^2}}\, dx \qquad r=1,\ R=2$$

$$\Rightarrow \boxed{A = 2\int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\, dx}$$

3. Write an integral that represents that area of the lune in the figure.



$y_2 = \sqrt{R^2 - r^2} + \sqrt{r^2 - x^2}$

$(r, \sqrt{R^2 - r^2})$

$y_1 = \sqrt{R^2 - x^2}$

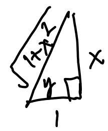
$$A = 2\int_0^r y_2 - y_1 \, dx$$

$$\boxed{A = 2\int_0^r \left(\sqrt{R^2 - r^2} + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2}\right) dx}$$

Part II: Proofs

Prove the formulas, showing enough work to justify each step.

4. $\dfrac{d}{dx}[\arctan(x)] = \dfrac{1}{1+x^2}$



$y = \tan^{-1} x \Rightarrow x = \tan y$

$\dfrac{d}{dx}\left[x = \tan y\right] \Rightarrow 1 = \sec^2 y \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \cos^2 y = \left(\dfrac{1}{\sqrt{1+x^2}}\right)^2 = \dfrac{1}{1+x^2}$

So $\dfrac{d}{dx}\left[\tan^{-1}(x)\right] = \dfrac{1}{1+x^2}$ ∎

5. $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$

$y = \cosh^{-1}(x) \Rightarrow x = \cosh(y) = \frac{1}{2}\left(e^y + e^{-y}\right)$

$\Rightarrow \left(e^y - 2x + e^{-y} = 0\right)e^y$

$\Rightarrow (e^y)^2 - 2x(e^y) + 1 = 0$

$(e^y)^2 - 2x(e^y) + x^2 = x^2 - 1$

$(e^y - x)^2 = x^2 - 1$

$e^y = x \pm \sqrt{x^2-1}$

$+$

So, $y = \cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$ ∎