§12.3

HW. Find a unit vector $\vec{u}$ that is orthogonal to both $\vec{w}_1 = \vec{i} + \vec{j}$ and $\vec{w}_2 = \vec{i} + \vec{k}$.

$\vec{u} = \langle x, y, z \rangle$ ← unknown vector TBD

$\vec{w}_1 = \langle 1, 1, 0 \rangle$

$\vec{w}_2 = \langle 1, 0, 1 \rangle$

$\|\vec{u}\| = 1 \Rightarrow \sqrt{x^2+y^2+z^2}^{\,2} = 1^2 \Rightarrow x^2+y^2+z^2 = 1$

orthogonality:

$\vec{u} \cdot \vec{w}_1 = 0 \Rightarrow \langle x,y,z\rangle \cdot \langle 1,1,0\rangle = x+y = 0$

$\vec{u} \cdot \vec{w}_2 = 0 \Rightarrow \langle x,y,z\rangle \cdot \langle 1,0,1\rangle = x+z = 0$

Now, solve the system

$$\begin{cases} x^2+y^2+z^2 = 1 \\ x+y=0 \quad \to \quad y = -x \\ x+z=0 \quad \to \quad z=-x \end{cases}$$

$3x^2 = 1 \Rightarrow x = \pm\sqrt{1/3}$ . . .

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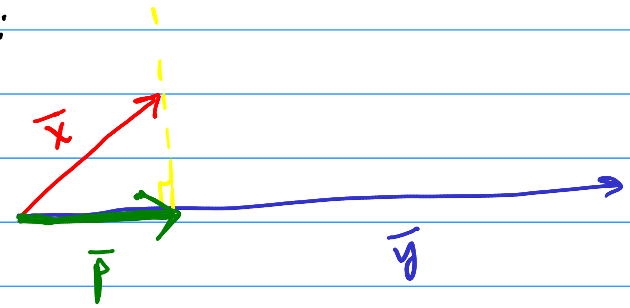
§12.3, cont'd.

$\bar{x}, \bar{y} \neq \bar{0}$




$\bar{p}$ is the (vector) projection of $\bar{x}$ onto $\bar{y}$.

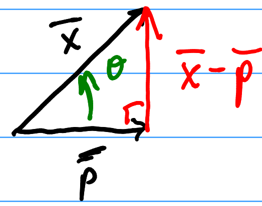
$\bar{p} = \mathrm{proj}_{\bar{y}}\bar{x}$

Q. Find a formula for $\bar{p}$.

$\bar{p}$ is in the same direction as $\bar{y}$.

$$\bar{p} = \alpha \frac{\bar{y}}{\|\bar{y}\|} \quad \text{so that} \quad |\alpha| = \|\bar{p}\|$$

Now, find $\alpha$.




Two approaches:

1. Apply pythag. thm, and try to solve for $\|\bar{p}\|$.
2. Use the formula for $\cos\theta$.

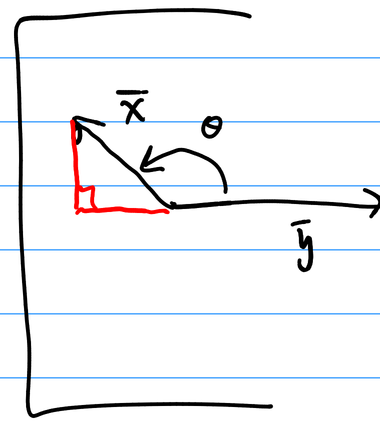
So, $\cos\theta = \dfrac{\bar{x}\cdot\bar{y}}{\|\bar{x}\|\,\|\bar{y}\|}$ from yesterday.

and $\cos\theta = \dfrac{\|\bar{p}\|}{\|\bar{x}\|}$ by Right Triangle Trig.

Setting these equal,

$$\frac{\|\bar{p}\|}{\cancel{\|\bar{x}\|}} = \frac{\bar{x}\cdot\bar{y}}{\cancel{\|\bar{x}\|}\,\|\bar{y}\|} \quad \text{and}$$

$$|\alpha| = \|\bar{p}\| = \frac{\bar{x}\cdot\bar{y}}{\|\bar{y}\|}$$




__Def'n.__ Let $\bar{x}, \bar{y} \neq \bar{0}$. The projection of $\bar{x}$ onto $\bar{y}$ is the vector,

$$(*) \quad \boxed{\operatorname{proj}_{\bar{y}} \bar{x} = \frac{\bar{x}\cdot\bar{y}}{\|\bar{y}\|}\,\frac{\bar{y}}{\|\bar{y}\|}}$$

or $$\bar{p} = \frac{\bar{x}\cdot\bar{y}}{\|\bar{y}\|^2}\,\bar{y} = \frac{\bar{x}\cdot\bar{y}}{\bar{y}\cdot\bar{y}}\,\bar{y}$$

__Ex.__ $\bar{x} = \langle 1,1,2\rangle \qquad \bar{y} = \langle -2,3,1\rangle$

$\operatorname{proj}_{\bar{y}}\bar{x}$ and $\operatorname{proj}_{\bar{x}}\bar{y}$

$$\operatorname{proj}_{\bar{y}}\bar{x} = \frac{\bar{x}\cdot\bar{y}}{\bar{y}\cdot\bar{y}}\,\bar{y} \qquad \operatorname{proj}_{\bar{x}}\bar{y} = \frac{\bar{y}\cdot\bar{x}}{\bar{x}\cdot\bar{x}}\,\bar{x}$$

$\bar{x}\cdot\bar{y} = -2+3+2 = 3$ $\qquad$ $\text{proj}_{\bar{y}}\bar{x} = \frac{3}{14}\langle -2,3,1\rangle = \langle \frac{-6}{14}, \frac{9}{14}, \frac{3}{14}\rangle$

$\bar{y}\cdot\bar{y} = 4+9+1 = 14$

$\bar{x}\cdot\bar{x} = 1+1+4 = 6$ $\qquad$ $\text{proj}_{\bar{x}}\bar{y} = \frac{1}{2}\langle 1,1,2\rangle = \langle \frac{1}{2}, \frac{1}{2}, 1\rangle$

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**Theorem.** Cauchy–Schwarz–Bunyachevsky Inequality (CSB ≠)

Let $\bar{x}$ and $\bar{y}$ be any vectors in $\mathbb{R}^n$ $(n=2,3)$. Then

$$|\bar{x}\cdot\bar{y}| \leq \|\bar{x}\|\|\bar{y}\|$$

**Proof.** Suppose $\bar{y}=\bar{0}$, then $\bar{x}\cdot\bar{y}=0$ and $\|\bar{y}\|=0$, so the two sides are equal.

Now suppose $\bar{x},\bar{y}\neq\bar{0}$.

Then

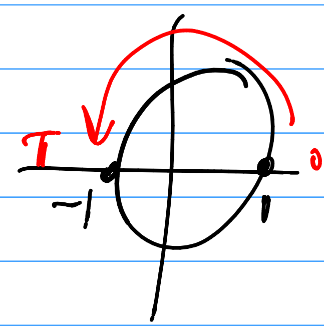
$$\bar{x}\cdot\bar{y} = \|\bar{x}\|\|\bar{y}\|\cos\theta$$

$$|\bar{x}\cdot\bar{y}| = \left|\|\bar{x}\|\|\bar{y}\|\cos\theta\right| = \|\bar{x}\|\|\bar{y}\|\,|\cos\theta|$$

but $\quad -1\leq\cos\theta\leq 1 \Rightarrow |\cos\theta|\leq 1$

then,

$$|\bar{x}\cdot\bar{y}| \leq \|\bar{x}\|\|\bar{y}\| \quad \blacksquare$$




**Q.** When $|\bar{x}\cdot\bar{y}| = \|\bar{x}\|\|\bar{y}\|$ ?

**A1.** if one is $\bar{0}$.

**A2.** if $\cos\theta=\pm 1 \Rightarrow \theta = 0, \pi$

if and only if $\bar{x}$ and $\bar{y}$ are parallel.

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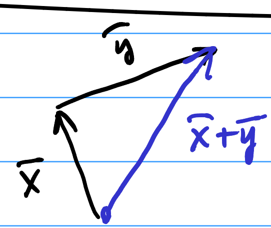
**Thm.** Triangle Inequality




$$\|\bar{x}+\bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\| \qquad \text{for all } \bar{x},\bar{y}.$$

Proof. $\left(\|\bar{x}+\bar{y}\|\right)^2 \overset{?}{\leq} \left(\|\bar{x}\| + \|\bar{y}\|\right)^2$

$$\|\bar{x}+\bar{y}\|^2 \overset{?}{\leq} \|\bar{x}\|^2 + 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2$$

Now, we'll work only on LHS:

$$\begin{aligned}
\|\bar{x}+\bar{y}\|^2 &= (\bar{x}+\bar{y})\cdot(\bar{x}+\bar{y}) \\
&= \bar{x}\cdot\bar{x} + 2\,\bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{y} \\
&= \|\bar{x}\|^2 + 2\,\bar{x}\cdot\bar{y} + \|\bar{y}\|^2 \\
&\leq \|\bar{x}\|^2 + 2|\bar{x}\cdot\bar{y}| + \|\bar{y}\|^2 && \text{by abs. value.} \\
&\leq \|\bar{x}\|^2 + 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2 && \text{by CSB} \neq \\
&= (\|\bar{x}\| + \|\bar{y}\|)^2. \quad \blacksquare
\end{aligned}$$