Let $\bar{x}, \bar{y} \in \mathbb{R}^n$.

The vector projection of $\bar{x}$ onto $\bar{y}$ is:

$$\text{proj}_{\bar{y}} \bar{x} = \underbrace{\frac{\bar{x} \cdot \bar{y}}{\|\bar{y}\|}}_{\text{scalar projection : } \text{comp}_{\bar{y}} \bar{x}} \frac{\bar{y}}{\|\bar{y}\|}$$

$$\|\text{proj}_{\bar{y}} \bar{x}\| = |\text{comp}_{\bar{y}} \bar{x}|$$

§12.4 Cross Product

only defined in $\mathbb{R}^3$!

Let $\bar{x} = \langle x_1, x_2, x_3 \rangle$ and $\bar{y} = \langle y_1, y_2, y_3 \rangle$

$$\circledast \quad \boxed{\bar{x} \times \bar{y} = \langle x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1 \rangle}$$

This can also be defined using the determinant of a 3x3 matrix.

$$\circledast \quad \bar{x} \times \bar{y} = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \vec{i}(x_2 y_3 - x_3 y_2) - \vec{j}(x_1 y_3 - x_3 y_1) + \vec{k}(x_1 y_2 - x_2 y_1)$$

$$= \langle x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1 \rangle$$

Ex. $\vec{a} = \langle 1, 3, 4 \rangle \quad \vec{b} = \langle 2, 7, -5 \rangle$

$\vec{a} \times \vec{b}$, $\vec{b} \times \vec{a}$ $\leftarrow$ Find both.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i}(-43) - \vec{j}(-13) + \vec{k}(1) = \langle -43, 13, 1 \rangle$$

$$\vec{b}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 7 & -5 \\ 1 & 3 & 4 \end{vmatrix} = \vec{i}(43) - \vec{j}(13) + \vec{k}(-1) = \langle 43, -13, -1 \rangle$$

<u>Thm.</u> $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ for all $\vec{a},\vec{b} \in \mathbb{R}^3$.

<u>Thm.</u> $\vec{a}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + 0\vec{k} = \langle 0,0,0\rangle = \vec{0}$.

$\vec{a}\times\vec{a} = \vec{0}$

<u>Thm.</u> $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

<u>Proof.</u> $(\vec{a}\times\vec{b})\cdot\vec{a} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle \cdot \langle a_1, a_2, a_3\rangle$

$$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1} = 0$$

Check: $(\vec{a}\times\vec{b})\cdot\vec{b} = 0$.

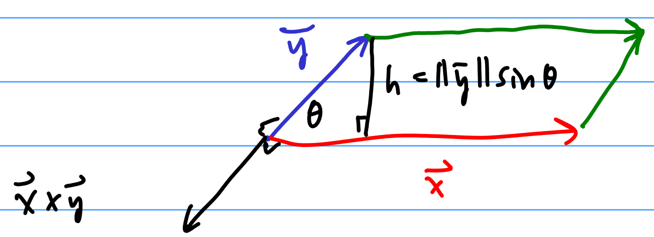
<u>Thm.</u> Let θ be the angle between the nonzero vectors $\vec{u}, \vec{v}$.
$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$. $\leftarrow (*)$

<u>Proof.</u> Write out $\|\vec{u}\times\vec{v}\|^2$ in terms of the dot product, then apply trig. ∎

<u>Corollary.</u> if $\vec{u}\times\vec{v} = \vec{0}$, then $\vec{u}$ and $\vec{v}$ are parallel.

$\theta = 0$ or $\theta = \pi$.

<u>Geometric Significance</u>



Area of ▱:
$A = \|\vec{x}\|\,\|\vec{y}\|\sin\theta$
so $\|\vec{x}\times\vec{y}\|$ is the area of the parallelogram!

Ex. $P(1,4,6)$ $Q(-2,5,-1)$ $R(1,-1,1)$

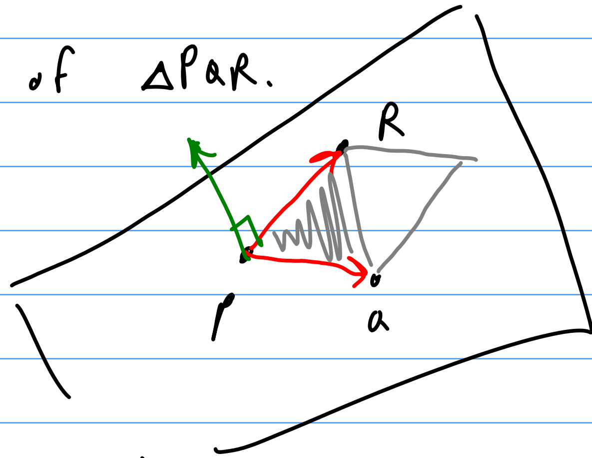
a.) Find a vector that is perpendicular to the plane through P, Q, R.

then,

b.) Find the area of ΔPQR.

$\overrightarrow{PQ} = \langle -3, 1, -7 \rangle$

$\overrightarrow{PR} = \langle 0, -5, -5 \rangle$



a.) $$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \vec{i}(-5-35) - \vec{j}(15-0) + \vec{k}(15-0)$$

$$= \langle -40, -15, 15 \rangle$$

b). Area of ΔPQR? $= \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

$$= \frac{1}{2}\sqrt{40^2 + 15^2 + 15^2} = \frac{5}{2}\sqrt{8^2+3^2+3^2}$$

$$= \boxed{\frac{5}{2}\sqrt{82}}$$

Properties. Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ and $k \in \mathbb{R}$. Then

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2. $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$