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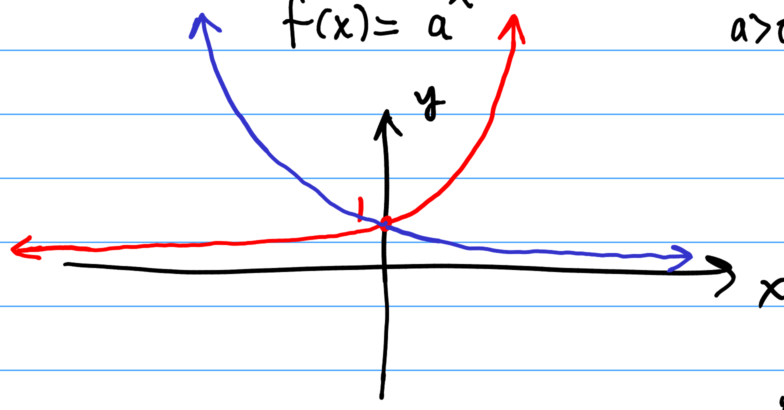
§6.1–6.4 – Exponential functions and logarithms

Def'n. An exponential function is a function of the form:

$$f(x) = a^x \qquad a>0,\ a \neq 1$$



$a > 1$

$0 < a < 1$

$\mathrm{dom}(f) = \mathbb{R}$

if $a = \frac{1}{b}$ for $b > 1$,
$a^x = \left(\frac{1}{b}\right)^x = b^{-x}$

Def'n. The natural number is $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx 2.718281828\ldots$?

The natural exponential function is:

$$\boxed{f(x) = e^x \qquad f'(x) = e^x}$$ ⊛

similarly, $\boxed{\int e^x\,dx = e^x + C}$

Given f, an antiderivative of is a function F satisfying

$$F' = f$$

Fundamental Theorem of Calculus (FTC)

1. $\frac{d}{dx}\int_a^x f(u)\,du = f(x)$
2. $\int_a^b \frac{d}{dx}[F(x)]\,dx = F(b) - F(a)$

Ex. $f(x) = 2^x$ ~~Pause.~~

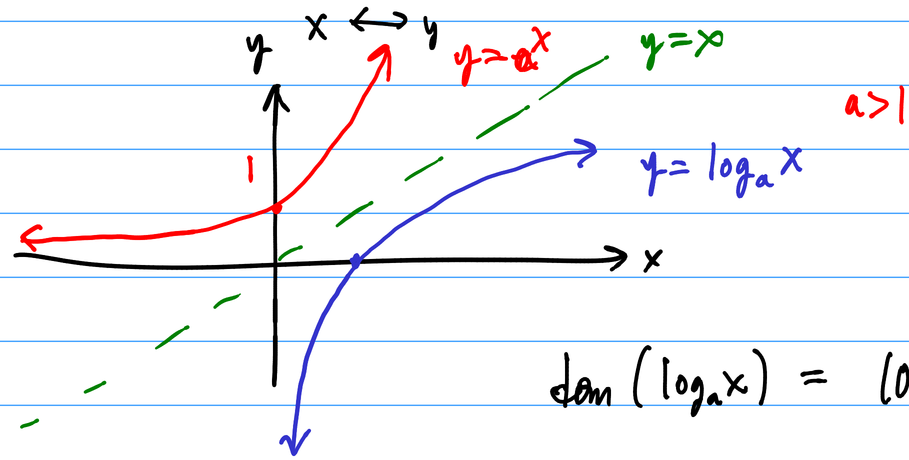
$(2^x)'$ $\qquad 2^x = \left(e^{\ln 2}\right)^x = e^{x\ln 2} = e^u$

$$(2^x)' = (e^u)' = u'e^u = \ln 2\left(e^{x\ln 2}\right) = \ln(2)\cdot 2^x$$

In general, $\boxed{(a^x)' = \ln(a)\cdot a^x}$

Def'n. $y = \log_a(x)$ means $a^y = x$ $\qquad a>0,\ a\neq 1$

logarithms are the inverse functions of exponentials!



$\text{dom}(\log_a x) = (0,\infty)$

Inverse relationships:

$$\begin{cases} \log_a(a^x) = x & \text{for all } x\in\mathbb{R} \\ a^{\log_a x} = x & \text{for } x>0 \end{cases}$$

$\log_e x$ is the natural log: $\log_e x = \ln(x)$

$$\begin{cases} \ln(e^x) = x & \\ e^{\ln x} = x & x>0 \end{cases}$$

Ex. $\frac{d}{dx}[\ln x] = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx \neq \frac{x^0}{0}$$

$$\int \frac{1}{x}\,dx = \ln(x) + C$$

Ex. $\frac{d}{dx}[\log_2 x] = \frac{d}{dx}\left[\frac{\ln x}{\ln 2}\right] = \frac{1}{\ln 2}\frac{d}{dx}[\ln x] = \frac{1}{x \ln 2}$

$$\boxed{\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}}$$

$a>0,\ a\neq 1$

Ex. $\int \ln x\,dx = ?$ can't do yet.

Ex. $y = x^x$ compute y' ?

$\ln(y = x^x)$

$\ln y = \ln(x^x) = x\ln(x)$

logarithmic differentiation.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln(x))$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1$$

$$\frac{dy}{dx} = y\cdot(\ln x + 1)$$

$$\boxed{\frac{dy}{dx} = x^x\ln x + x^x} = (x^x)'$$

Next, Build new functions from these. Do Calculus.

Hyperbolic Trig Functions:

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}) \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$$

Find $\cosh'(x)$, $\sinh'(x)$, $\tanh'(x)$