M243 9/4

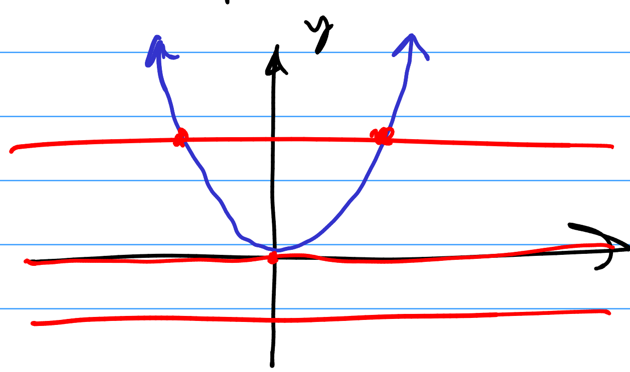
§6.6 – Inverse Trig Functions

Let f be a one-to-one function:

b/c it's a function: for every input there is exactly one output

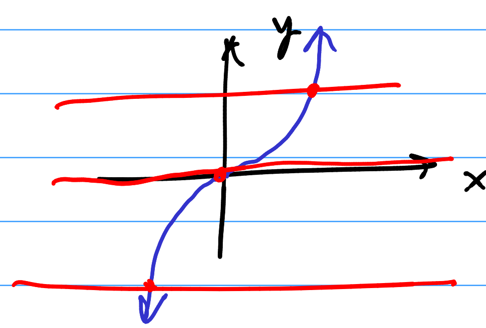
one-to-one: for every output there is ~~only~~ exactly one input

Ex. $f(x) = x^2$



NOT 1-1

Ex. $f(x) = x^3$



$y = x^3$

1-1 !

If f is one-to-one, then f has an inverse function,

$y = f(x) \longrightarrow x = f(y)$

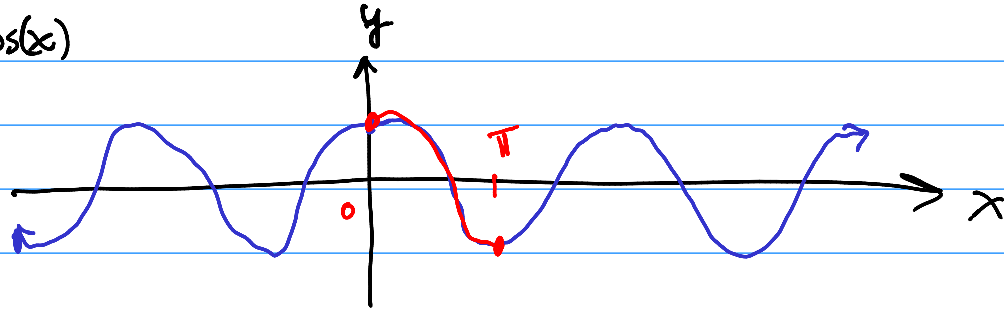
$y = f^{-1}(x)$ for this function

Ex. $y = x^3$ is 1-1. $\sqrt[3]{x} = \sqrt[3]{y^3}$ then $y = f^{-1}(x) = \sqrt[3]{x}$.

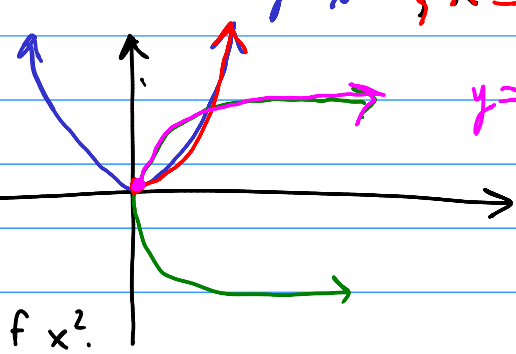
Ex. $y = \cos(x)$

Fails HLT

Not 1-1.



Consider $f(x) = x^2$.

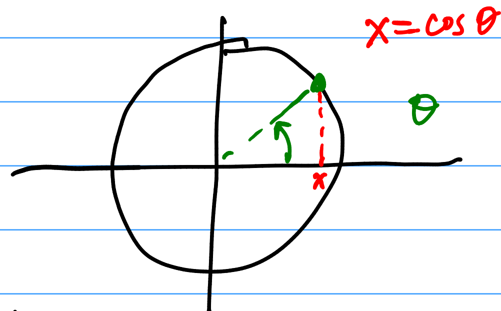


To make an inverse function, we need to restrict the domain of x^2.

Ex. if $f(x) = \cos(x)$, $0 \le x \le \pi$
then f is 1-1, and hence has an inverse function.

$$\underbrace{y = \cos^{-1}(x) \quad \underline{\underline{\text{means}}} \quad x = \cos(y), \quad 0 \le y \le \pi}$$

Unit Circle:



Restriction:
$0 \le \theta \le \pi$: top half

$\cos^{-1}(x) = \theta$

$\cos^{-1}(x) = \arccos(x) = \theta$

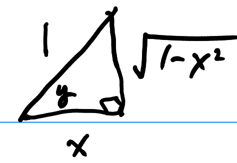
Ex. $\frac{d}{dx}\left[\arccos(x)\right]$ $\qquad y = \arccos(x)$

$\frac{d}{dx}\left[\cos(y) = x\right]$ $\qquad$ Implicit

$-\sin(y) \cdot \frac{dy}{dx} = 1$ $\qquad$ Chain Rule!

Then, $\dfrac{dy}{dx} = \dfrac{-1}{\sin(y)}$

$\cos(y) = \dfrac{x}{1}$

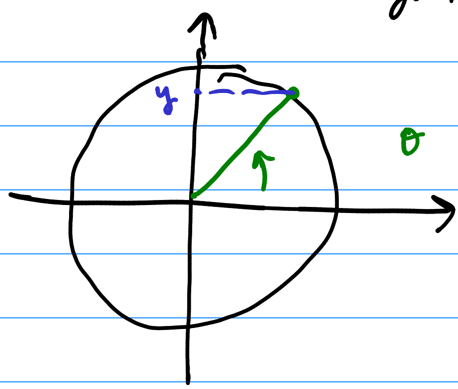


$$\boxed{\frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}}$$

$1^2 = x^2 + b^2$

$b = \sqrt{1-x^2}$

Ex. arcsine and arctangent



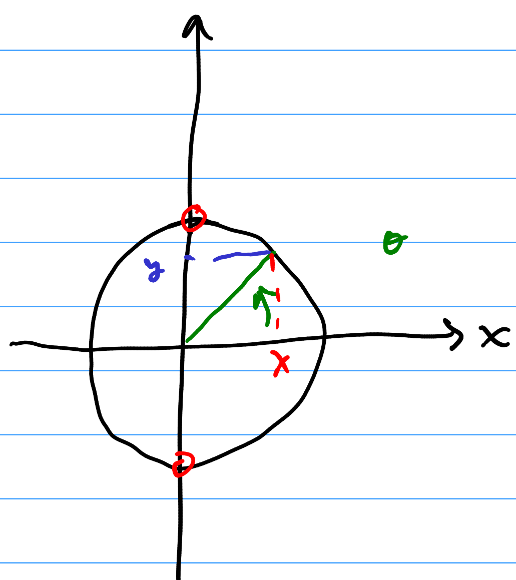
$\arcsin(y) = \theta$

Restriction : RHS:

Range of arcsin $\Big\}$: $-\dfrac{\pi}{2} \leq \theta \leq \pi/2$

Domain? $-1 \leq x \leq 1$

$$y = \sin^{-1}(x) \iff \sin(y) = x, \quad -\pi/2 \leq y \leq \pi/2$$



$\tan\theta = \underbrace{y/x}_{u}$ $\left\{ \begin{array}{l} -\infty < u < \infty \\ -\pi/2 < \theta < \pi/2 \end{array} \right.$

$\theta = \arctan(u)$

$$\boxed{y = \arctan(x) \iff x = \tan(y), \quad -\pi/2 < y < \pi/2}$$

Domain of $\tan^{-1}$ is $\mathbb{R}$.

Ex. (CRE) $\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$

$1-x^2>0$

$x^2<1$

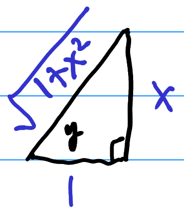
$|x|<1$

$\underline{-1<x<1}$

Ex. $\frac{d}{dx}\left[\tan^{-1}(x)\right]$

$y=\tan^{-1}(x)$

$\frac{d}{dx}\left[\underline{\tan(y)}=x\right]$



$\sec^2(y)\cdot\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{\sec^2(y)}=\cos^2(y)=\left(\cos(y)\right)^2$

$c^2=1^2+x^2$

$c=\sqrt{1+x^2}$

$=\left(\frac{1}{\sqrt{1+x^2}}\right)^2=\frac{1}{1+x^2}$

$$\boxed{\frac{d}{dx}\left[\arctan(x)\right]=\frac{1}{1+x^2}}$$ (*)

Ex. $\int\frac{1}{1+x^2}\,dx=\int\frac{d}{dx}\left[\arctan(x)\right]dx=\arctan(x)+C$

$\cancel{u=1+x^2}$

$\cancel{du=2x\,dx}$

Ex. $\int\frac{1}{\sqrt{1-x^2}}\,dx=\int\frac{d}{dx}\left[\arcsin(x)\right]dx=\arcsin(x)+C_1$

OR $\int\frac{d}{dx}\left[-\arccos(x)\right]dx=-\arccos(x)+C_2$

$\underline{\text{Ex}}$ $\displaystyle\int \frac{2x}{1+x^4}\,dx = \int \frac{\overbrace{2x\,dx}^{du}}{1+(x^2)^2} = \int \frac{1}{1+u^2}\,du$

$\cancel{u = 1+x^4}$

$\cancel{du = 4x^3\,dx}$

$u = x^2$

$du = 2x\,dx$

$= \arctan(u) + C$

$= \arctan(x^2) + C$
