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§6.6 – $(\arctan x)' = \frac{1}{x^2+1}$

Ex. $\int \frac{1}{x^2+a^2}\,dx$, $a>0$

our job: find $F(x)$ s.t. $F'(x) = \frac{1}{x^2+a^2}$

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2\left(\frac{x^2}{a^2}+1\right)}\,dx = a\frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,\underbrace{\frac{1}{a}dx}_{du}$$

let $u = \frac{x}{a}$

$du = \frac{1}{a}\,dx \rightarrow a\,du = dx$

$$= \frac{1}{a}\int \frac{1}{u^2+1}\,du = \frac{1}{a}\arctan(u) + C$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C} \quad (*)$$

Ex. $\frac{1}{2}\int \frac{2x}{x^4+9}\,dx$

try

~~$u = x^4+9$~~

~~$du = 4x^3\,dx$~~

$x^4 = (\underbrace{x^2}_{u})^2$

let $u = x^2$

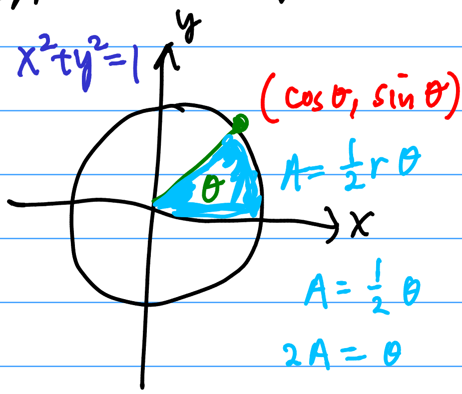
$du = 2x\,dx$

$$= \underbrace{\frac{1}{2}\int \frac{1}{u^2+9}\,du}_{\text{use formula.}} = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right) + C$$

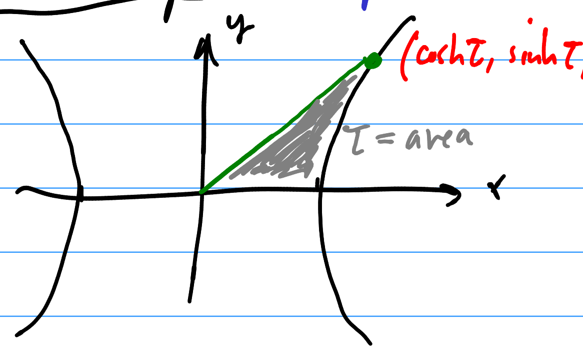
$$= \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C$$

§6.7 – Hyperbolic Trig. Functions

Trig:



Hyperbolic Trig.



We get some nice formulas:

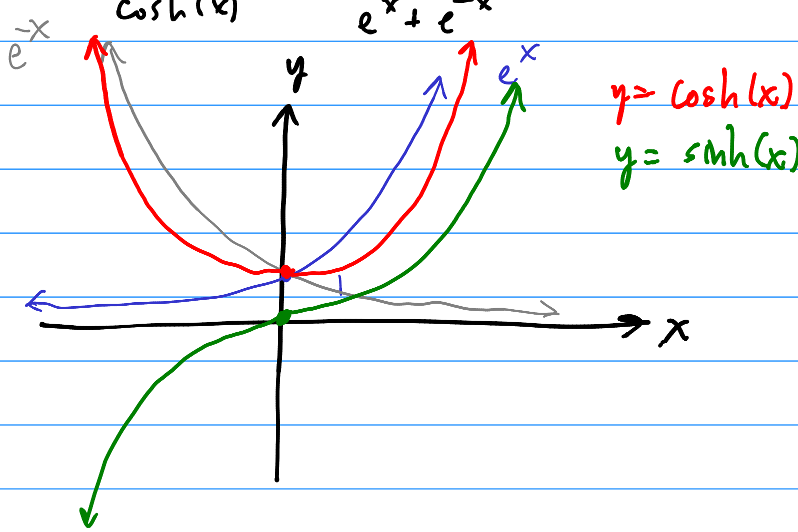
$$\begin{cases} \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \\ \sinh(x) = \frac{1}{2}(e^x - e^{-x}) \end{cases}$$

$$\rightarrow \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\coth(x) = \frac{1}{\tanh(x)}$$



cosh is an even function: $\cosh(-x) = \cosh(x)$

sinh is odd: $\sinh(-x) = -\sinh(x)$

Hyperbolic Pythagorean Theorem – $\cosh^2 x - \sinh^2 x = 1$ for all x.

Check:

$$\cosh^2 x = \left(\tfrac{1}{2}(e^x + e^{-x})\right)^2 = \tfrac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$$

$$- \quad \sinh^2 x = \left(\tfrac{1}{2}(e^x - e^{-x})\right)^2 = \tfrac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}\left(0 + 4 + 0\right) = \tfrac{1}{4}\cdot 4 = 1$$

Calculus:

$$\frac{d}{dx}\left[\cosh(x)\right] = \frac{d}{dx}\left[\tfrac{1}{2}(e^x + e^{-x})\right]$$

$$= \tfrac{1}{2}\left[\frac{d}{dx}[e^x] + \frac{d}{dx}[e^{-x}]\right]$$

$$= \tfrac{1}{2}\left(e^x - e^{-x}\right)$$

$$= \sinh(x)$$

$$\frac{d}{dx}\left[\sinh(x)\right] = \frac{d}{dx}\left(\tfrac{1}{2}(e^x - e^{-x})\right)$$

$$= \tfrac{1}{2}\left(e^x + e^{-x}\right)$$

$$= \cosh(x)$$

$$\frac{d}{dx}\left(\tanh(x)\right) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\cosh x)(\sinh x)' - (\sinh x)(\cosh x)'}{(\cosh x)^2}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\int \sinh(x)\,dx = \cosh x + C \quad , \quad \int \operatorname{sech}^2 x\,dx = \tanh x + C$$

Inverse Hyperbolic Trig functions.

$y = \sinh^{-1}(x)$ means $\sinh(y) = x$

Find a formula for y:

$$x = \sinh(y) = \frac{1}{2}(e^y - e^{-y})$$

$$e^y\left(2x = e^y - e^{-y}\right)$$

$$2xe^y = (e^y)^2 - 1$$

$$\underbrace{(e^y)}_{u}{}^2 - 2x\underbrace{e^y}_{u} - 1 = 0$$

$$u^2 - (2x)u - 1 = 0$$

$$\left(u^2 \underbrace{- 2x}_{b}\, u + x^2\right) = 1 + x^2$$

$b = -2x$

$\frac{b}{2} = -x$

$\left(\frac{b}{2}\right)^2 = x^2$

$$(u - x)^2 = 1 + x^2$$

$$u - x = \pm\sqrt{1+x^2}$$

$$e^y = u = x \pm \sqrt{1+x^2}$$

$$\ln\left(e^y = x + \sqrt{1+x^2}\right)$$

$$\boxed{\sinh^{-1}(x) = y = \ln\left(x + \sqrt{1+x^2}\right)}$$