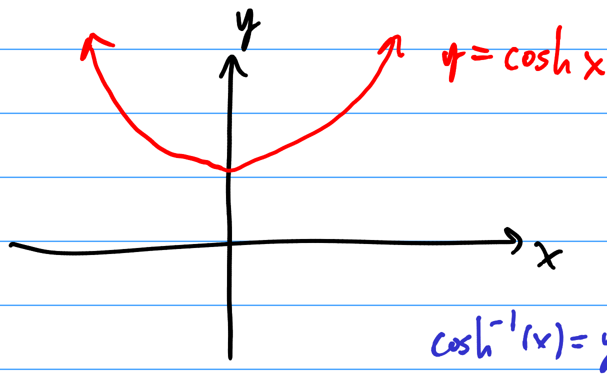
M243 9/6/18

§6.7 - Hyperbolic Trig

yesterday: $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$, $x \in \mathbb{R}$



$y \geq 1$

$\cosh^{-1}(x) = y \Rightarrow x = \cosh(y)$

$x \geq 1$

Find a formula for $\cosh^{-1}(x)$:

$\frac{1}{2}\left(e^{y} + e^{-y}\right) = x$ solve for y.

$e^{y} + e^{-y} = 2x$

$e^{y}\left(e^{y} - 2x + e^{-y} = 0\right)$

$\left(e^{y}\right)^2 - 2x e^{y} + 1 = 0$

$e^{y} = \dfrac{2x \pm \sqrt{4x^2 - 4\cdot1\cdot1}}{2\cdot1}$

$= \dfrac{2x \pm 2\sqrt{x^2-1}}{2}$

$+$

$e^{y} = x \pm \sqrt{x^2-1}$

$\ln\left(e^{y} = x + \sqrt{x^2-1}\right)$

$y = \boxed{\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)}$ $x \geq 1$

Ex. $\boxed{\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$ ⊛

$$\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$$
$$\cosh^{-1}(x) = \ln\left(x+\sqrt{x^2-1}\right)$$

$$(\ln u)' = \frac{u'}{u} \qquad (\sqrt{u})' = \frac{u'}{2\sqrt{u}}$$

$$\left(\sinh^{-1}(x)\right)' = \left(\ln\left(x+\sqrt{x^2+1}\right)\right)' = \frac{u'}{u} = \frac{1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$$

$$= \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\frac{\cancel{\left(\sqrt{x^2+1}+x\right)}}{\sqrt{x^2+1}}}{\frac{x+\sqrt{x^2+1}}{1}} \cdot \frac{1}{\cancel{\left(x+\sqrt{x^2+1}\right)}}$$

$$\boxed{\left(\sinh^{-1}(x)\right)' = \frac{1}{\sqrt{x^2+1}}}$$ ⊛

Ex. $$\left(\cosh^{-1}(x)\right)' = \left(\ln\left(x+\sqrt{x^2-1}\right)\right)'$$

$$= \frac{1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{\frac{\cancel{\left(\sqrt{x^2-1}+x\right)}}{\sqrt{x^2-1}}}{\frac{x+\sqrt{x^2-1}}{1}} \cdot \frac{1}{\cancel{\left(x+\sqrt{x^2-1}\right)}}$$

$$\boxed{\frac{d}{dx}\left[\cosh^{-1}(x)\right] = \frac{1}{\sqrt{x^2-1}}}$$ ⊛

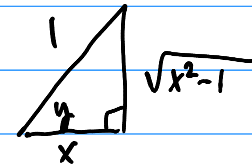
$$\boxed{\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}}$$

Ex. Another way: $\left(\cosh^{-1}(x)\right)'$

$y=\cosh^{-1}(x)$ means $\cosh(y)=x$ $\qquad y=\sqrt{x^2-1}$

$\frac{d}{dx}\left[\cosh(y)=x\right] \rightarrow \sinh(y)\cdot\frac{dy}{dx}=1$ $\qquad x^2-y^2=1$

$$\frac{dy}{dx}=\frac{1}{\sinh(y)}$$



$$\frac{dy}{dx}=\left(\cosh^{-1}(x)\right)'=\frac{1}{\sqrt{x^2-1}}$$

$\sinh(y)=\frac{O}{H}=\frac{x}{1}$

Ex. $\tanh^{-1}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=\frac{1}{2}\left(\ln(1+x)-\ln(1-x)\right)$

$$\left(\tanh^{-1}(x)\right)'=\frac{1}{2}\left(\frac{1-x}{(1+x)(1-x)}+\frac{1+x}{(1-x)(1+x)}\right)=\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{1-x^2}\right)$$

$$\boxed{\frac{d}{dx}\left[\tanh^{-1}(x)\right]=\frac{1}{1-x^2}}$$ (*)

Now we can integrate more stuff!

$\int\frac{1}{\sqrt{x^2+1}}\,dx=\sinh^{-1}(x)+C$	$\int\frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}(x)+C$
$\int\frac{1}{\sqrt{x^2-1}}\,dx=\cosh^{-1}(x)+C$	$\int\frac{1}{1+x^2}\,dx=\tan^{-1}(x)+C$
$\int\frac{1}{1-x^2}\,dx=\tanh^{-1}(x)+C$	

$\underline{Ex}.\ \int_4^6 \frac{1}{\sqrt{t^2-9}}\,dt = \int_4^6 \frac{1}{3\sqrt{\left(\frac{t}{3}\right)^2-1}}\,dt = \int \frac{1}{\cancel{3}\sqrt{u^2-1}}\,\cancel{3}\,du$

let $u = t/3 \leftarrow u(6) = \frac{6}{3} = 2$

$du = 1/3\,dt \qquad u(4) = 4/3$

$\underline{3du} = dt$

$$= \int_{4/3}^{2} \frac{1}{\sqrt{u^2-1}}\,du = \cosh^{-1}(u)\Big|_{u=4/3}^{2} = \cosh^{-1}(2) - \cosh^{-1}(4/3)$$

$$\approx \frac{0.52\ldots}{?}$$