M243 9/11/18

§7.2 – Trig Integration

$\cos^2(x) = \cancel{\cos(\cos(x))}$ $(\cos x)^2$

Ex. $\int \cos^3 x\,dx = \int (\cos x)^3\,dx$

$\cancel{u = \cos x}$
$\cancel{du = -\sin x\,dx}$

$\cos^2 x + \sin^2 x = 1$
$\cos^2 x = 1 - \sin^2 x$

$= \int \cos^2 x \cdot \cos x\,dx$

$= \int (1 - \sin^2 x)\cos x\,dx$

$u = \sin x$
$du = \cos x\,dx$

$= \int (1-u^2)\,du = u - \frac{1}{3}u^3 + C$

$$\boxed{\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C}$$

Ex. $\int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x \sin x\,dx$

$= \int (\sin^2 x)^2 \cos^2 x \sin x\,dx$

$= -\int (1 - \cos^2 x)^2 \cos^2 x\,(-\sin x\,dx)$

$u = \cos x$
$du = -\sin x\,dx$

$= -\int (1-u^2)^2 u^2\,du$

$= -\int (u^4 - 2u^2 + 1)u^2\,du$

$= -\int u^6 - 2u^4 + u^2\,du$

$$= -\left(\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right) + C$$

$$\boxed{\int \sin^5 x \cos^2 x \, dx = -\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$$

Ex. $\int_0^{\pi} \sin^2 x \, dx = \frac{1}{2}\int_0^{\pi} 1 - \cos(2x) \, dx$

$$\boxed{\begin{aligned} \cos^2 x &= \frac{1}{2}(1+\cos(2x)) \\ \sin^2 x &= \frac{1}{2}(1-\cos(2x)) \end{aligned}}$$

$u = 2x$ $u(\pi) = 2\pi$

$du = 2\,dx$ $u(0) = 0$

$$= \underbrace{\frac{1}{2}\int_0^{\pi} 1 \, dx}_{} - \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi} \cos(2x)\, 2\, dx$$

$$\frac{1}{2}x\Big|_0^{\pi} - \frac{1}{4}\int_0^{2\pi} \cos u \, du$$

$$\frac{1}{2}x\Big|_0^{\pi} - \frac{1}{4}\sin u\Big|_0^{2\pi}$$

$$\underbrace{\frac{1}{2}(\pi - 0)}_{} \cancel{- \frac{1}{4}(0 - 0)}$$

$$= \frac{1}{2}\pi$$

Ex. $\int \sin^5 x \, dx = \int \underbrace{\sin^4 x}_{} \sin x \, dx = \int \underbrace{(\sin^2 x)}_{\text{Pyth.}}{}^2 \sin x \, dx$

$$= -\int (1-\cos^2 x)^2 (-\sin x \, dx) = -\int (1-u^2)^2 \, du$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int u^4 - 2u^2 + 1 \, du$$

$$= -\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C$$

$$\int \sin^5 x \, dx = -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$$

Ex. $\int \sin^4 x \, dx = \int \sin^3 x \sin x \, dx$

$$= \int (\sin^2 x)^2 \, dx = \int \left(\tfrac{1}{2}(1-\cos(2x))\right)^2 dx$$

$$= \frac{1}{4}\int \underbrace{\cos^2(2x)}_{\frac{1}{2}(1+\cos(4x))} - 2\cos(2x) + 1 \, dx$$

$$= \frac{1}{4}\int \frac{1}{2} + \frac{1}{2}\cos(4x) - 2\cos(2x) + 1 \, dx$$

$$= \frac{1}{4}\int \frac{3}{2} + \frac{1}{2}\cos(4x) - 2\cos(2x) \, dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \frac{1}{2}\frac{1}{4}\sin(4x) - 2\frac{1}{2}\sin(2x)\right) + C$$

$$\boxed{\int \sin^4 x \, dx = \frac{3}{8}x + \frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + C}$$

Ex. $\int \tan^6 x \sec^4 x \, dx$

$du = \sec^2 x \, dx$

$u = \tan x$

$$= \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(1 + \tan^2 x) = \sec^2 x$$

$$= \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^6 (1+u^2) \, du = \int u^6 + u^8 \, du = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C$$

$$\int \tan^6 x \sec^4 x \, dx = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

Ex. $\int \tan^5\theta \sec^7\theta \, d\theta$

$u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$= \int \tan^4\theta \sec^6\theta \; \sec\theta \tan\theta \, d\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$= \int (\tan^2\theta)^2 \sec^6\theta \; \sec\theta \tan\theta \, d\theta$

$= \int (\sec^2\theta - 1)^2 \sec^6\theta \; \sec\theta \tan\theta \, d\theta$

$= \int (u^2-1)^2 u^6 \, du = \int (u^4 - 2u^2 + 1) u^6 \, du = \int u^{10} - 2u^8 + u^6 \, du$

$= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C$

$$\boxed{\int \tan^5\theta \sec^7\theta \, d\theta = \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C}$$