M243 9/12/18

$$\int \arccos(x)\,dx = u v - \int v\,du$$

$u = \arccos(x)$ $dv = dx$

$du = \frac{-1}{\sqrt{1-x^2}}\,dx$ $v = x$

$$\int \arccos(x)\,dx = x\arccos(x) + \frac{-1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$u = 1-x^2$

$du = -2x\,dx$

$$-\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\int u^{-1/2}\,du$$

$$-\frac{1}{\cancel{2}}\,u^{1/2}\cdot\cancel{2}$$

$$\boxed{\int \arccos(x)\,dx = x\cos^{-1}(x) - \sqrt{1-x^2} + C}$$

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§7.3 Integration by Trig. Subs

Ex. $\int \frac{1}{\sqrt{1-x^2}}\,dx$ $x = \sin\theta \leftarrow \ldots$

$dx = \cos\theta\,d\theta$

$$= \int \frac{1}{\underbrace{\sqrt{1-\sin^2\theta}}_{\cos^2\theta}}\cos\theta\,d\theta$$

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$= \int \frac{1}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta = \int 1\,d\theta = \theta + C$$

$$\text{(*)}\quad \boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C}$$

Table of substitutions

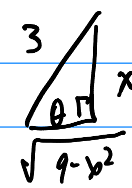
Expression	Subs	Domain	Why?
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2-a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2+a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \pi/2$, $\pi \le \theta < 3\pi/2$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

$a>0$

<u>Ex.</u> $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$9-x^2 = 3^2-x^2 \qquad a=3$

$$\begin{cases} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{cases}$$



$\frac{x}{3} = \sin\theta$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$$= \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2}\,3\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{9(1-\sin^2\theta)}\;3\cos\theta}{9\sin^2\theta}\,d\theta$$

$$= \int \frac{\sqrt{9\cos^2\theta}\;3\cos\theta}{9\sin^2\theta}\,d\theta = \int \frac{\cancel{3}\cos\theta\cdot\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int \csc^2\theta - 1\,d\theta$$

$$= -\cot\theta - \theta + C$$

Now get rid of the θs.

$$\boxed{\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C}$$

Ex. $\displaystyle\int \frac{1}{x^2\underbrace{\sqrt{x^2+4}}}\,dx$

$x^2+4 = x^2+2^2$

$\boxed{x = 2\tan\theta}$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{1 \cdot \cancel{2}\sec^2\theta}{\underset{2}{\cancel{4}}\tan^2\theta\sqrt{4\tan^2\theta+4}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{2\tan^2\theta\sqrt{4}\sqrt{1+\tan^2\theta}}\,d\theta = \frac{1}{4}\int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta\cdot\cancel{\sec\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

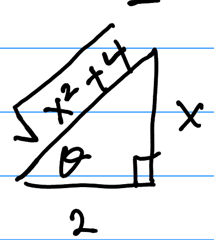
$u = \sin\theta$

$du = +\cos\theta\,d\theta$

$$= \frac{1}{4}\int u^{-2}\,du = -\frac{1}{4}u^{-1} + C$$

$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$



$$= -\frac{1}{4}\frac{1}{\sin\theta} + C$$

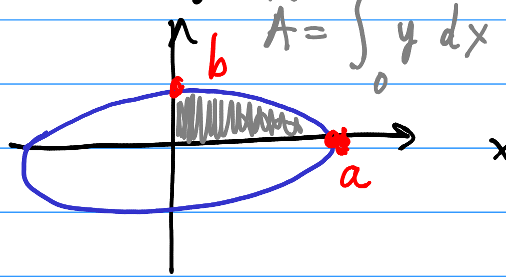
$$= -\frac{1}{4}\csc\theta + C$$

$$= \boxed{-\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + C}$$

Ex. Find the area of the ellipse $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$

$a,b>0$, $a\neq b$

$\hat{A}=\int_0^a y\,dx$



$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

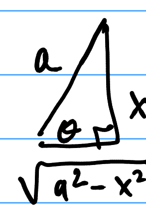
$$y=\sqrt{b^2-b^2\frac{x^2}{a^2}}=b\sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$A=4\hat{A}=4\int_0^a b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$\left.\begin{aligned} x&=a\sin\theta \leftarrow\\ dx&=a\cos\theta\,d\theta\end{aligned}\right\}=\frac{4b}{a}\int_{x=0}^{a} a\cos\theta\cdot a\cos\theta\,d\theta$$

$$=4ab\int_{x=0}^{a}\cos^2\theta\,d\theta=4ab\int_{x=0}^{a}\frac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$A=2ab\int_{x=0}^{a}1\,d\theta+2ab\int_{x=0}^{a}\cos(2\theta)\,d\theta$$



$$2ab\,\theta\Big|_{x=0}^{a}+\frac{2ab}{2}\sin(2\theta)\Big|_{x=0}^{a}$$

$\rightarrow \sin(2\theta)=2\sin\theta\cos\theta$

$$2ab\sin^{-1}\left(\frac{x}{a}\right)\Big|_{x=0}^{a}+ab\cdot 2\cdot\frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}\Big|_{x=0}^{a}$$

$$A=2ab\left(\sin^{-1}(1)-\sin^{-1}(0)\right)+\underbrace{2ab\left(\frac{a}{a}\frac{\sqrt{\cancel{a^2-a^2}}}{\cancel{a}}-\frac{0}{\cancel{a}}\frac{\sqrt{a^2-0^2}}{a}\right)}_{0=}$$

0 0

so,

$$A = 2ab\left(\pi/2 - 0\right) = \boxed{\pi ab}$$