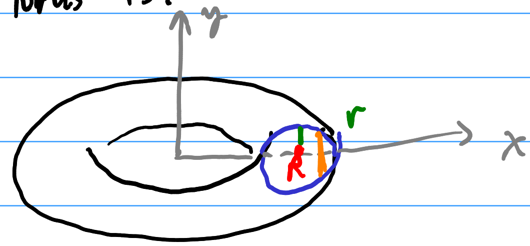
M243 9/13/18

§7.3 Trig. Subsitution

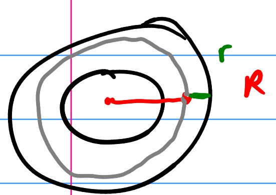
Ex. A torus is:

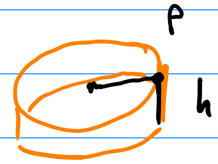


$$(x-h)^2 + (y-k)^2 = \rho^2$$

$$\underbrace{(x-R)^2 + y^2 = r^2}$$

From above



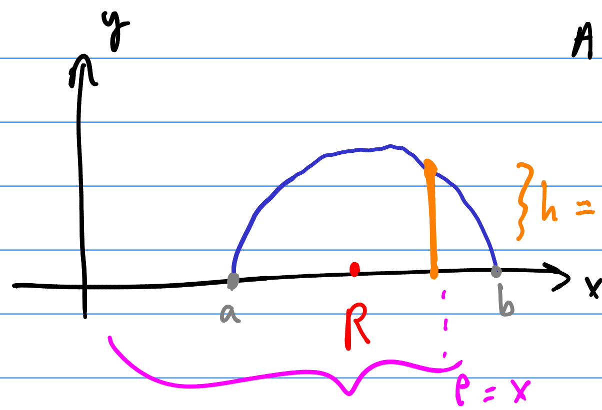


Goal: Find the volume.

use cylindrical shell method: $V = \int_a^b A\,dx$

$A = 2\pi \rho \cdot h$

Zoom in:



$h = y = \sqrt{r^2 - (x-R)^2}$

$a = R - r$

$b = R + r$

$$\tfrac{1}{2} V = \int_a^b A(x)\,dx = \int_a^b 2\pi \rho h\,dx = 2\pi \int_{R-r}^{R+r} x\sqrt{r^2-(x-R)^2}\,dx$$

$$V = 4\pi \int_{R-r}^{R+r} x\sqrt{r^2-(x-R)^2}\,dx = 4\pi \int_{-r}^{r} (u+R)\sqrt{r^2-u^2}\,du =$$

$u = x - R \rightarrow x = u + R$ $\qquad$ $u(R+r) = R + r - R = r$

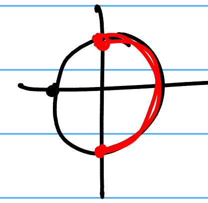
$du = dx$ $\qquad$ $u(R-r) = R - r - R = -r$

$$= 4\pi \int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi \int_{-r}^{r} R\sqrt{r^2-u^2}\,du$$

$$\underbrace{\qquad\qquad\qquad}_{I_1} \qquad \underbrace{\qquad\qquad\qquad}_{I_2}$$

$$V = 4\pi (I_1 + I_2)$$

$$I_1 = \frac{-1}{2}\int_{-r}^{r} -2u\sqrt{r^2-u^2}\,du = -\frac{1}{2}\int_0^0 \sqrt{v}\,dv = 0$$



$$v = r^2 - u^2 \qquad v(-r) = r^2 - (-r)^2 = r^2 - r^2 = 0$$
$$dv = -2u\,du \qquad v(r) = r^2 - r^2 = 0$$

$$I_2 = R\int_{-r}^{r} \sqrt{r^2-u^2}\,du$$

$$u = r\sin\theta \longleftrightarrow \sin\theta = \frac{u}{r} \rightarrow \theta = \sin^{-1}\left(\frac{u}{r}\right)$$
$$du = r\cos\theta\,d\theta$$
$$\theta(r) = \sin^{-1}\left(\frac{r}{r}\right) = \sin^{-1}(1) = \pi/2$$
$$\theta(-r) = \sin^{-1}\left(\frac{-r}{r}\right) = \sin^{-1}(-1) = -\pi/2$$

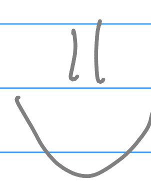
$$= R\int_{-\pi/2}^{\pi/2} \sqrt{r^2(1-\sin^2\theta)}\, r\cos\theta\,d\theta = R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta$$

$$= r^2R\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta = \frac{r^2R}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{r^2R}{2}\left[\pi/2 - (-\pi/2) + \underbrace{\cancel{\sin(\pi)}}_{0} - \underbrace{\cancel{\sin(-\pi)}}_{0}\right]$$

$$= \frac{r^2R}{2}\cdot\pi$$

$$V = 4\pi(I_1 + I_2) = 4\pi\cdot\frac{\pi r^2 R}{2} = 2\pi R\cdot\pi r^2$$



Ex. $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$ $\quad\begin{cases} a^2+u^2 \\ a^2-u^2 \\ u^2-a^2 \end{cases}$

$$3-2x-x^2$$
$$= -\left((x^2+2x+1) - 3 - 1\right)$$
$$= -\left[(x+1)^2 - 2^2\right]$$
$$= 2^2 - (x+1)^2$$

try: $\cancel{u = 3-2x-x^2}$

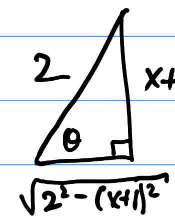
$\cancel{du = -2-2x} = -2(1+x)\,dx$

$$= \int \frac{x}{\sqrt{2^2-(x+1)^2}}\,dx$$

let $\quad x+1 = 2\sin\theta \qquad x = 2\sin\theta - 1 \qquad \sin\theta = \frac{x+1}{2}$

$\qquad dx = 2\cos\theta\, d\theta \qquad\qquad \theta = \sin^{-1}\left(\frac{x+1}{2}\right)$

$$= \int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}}\,\cancel{2\cos\theta}\,d\theta = \int 2\sin\theta - 1\,d\theta$$



$$= -2\cos\theta - \theta + C$$

$$= -\frac{2}{2}\sqrt{4-(x+1)^2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

$$\boxed{\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + C}$$