M243 9/14/18

§7.4 Partial Fraction Decomposition (PFD)

Any polynomial can be factored uniquely into terms of the form

$$(x-k) \quad \text{or} \quad (x^2+bx+c).$$

This means that any rational function $R(x) = \frac{P(x)}{Q(x)}$ w/ $\deg(P) < \deg(Q)$ can decomposed into a sum of the form

$$\frac{P(x)}{Q(x)} = \frac{A}{x-k} + \frac{B}{x-c} + \cdots + \frac{Cx+D}{x^2+bx+c} + \cdots$$

for a repeated factor:

$$\frac{x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$\frac{x^3-1}{(x^2+2x+2)^2} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}$$

Ex. $\displaystyle\int \frac{7t-5}{t+1}\,dt = \int 7 - \frac{12}{t+1}\,dt = \int 7\,dt - 12\int \frac{1}{t+1}\,dt$

$= 7t - 12\ln(t+1) + C$

long div:

$$\begin{array}{r} 7 + \frac{-12}{t+1} \\ t+1 \overline{\smash{)}\,7t-5} \\ -\ 7t+7 \\ \hline -12 \end{array}$$

Ex. $\int \frac{x-6}{x^2+x-6}\,dx$

$u = x^2+x-6$

$du = (2x+1)\,dx$

$x^2+x-6 = (x+3)(x-2)$

$$\frac{x-6}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$$ Find A, B

$$\frac{x-6}{x^2+x-6} = \frac{A(x-2)+B(x+3)}{x^2+x-6}$$

$$x-6 = A(x-2) + B(x+3)$$

$x=-3$: $\quad -9 = -5A + \cancel{0B}$ $\quad$ so $\quad A = 9/5$

$x=2$: $\quad -4 = \cancel{0A} + 5B$ $\quad\quad B = -4/5$

$$\int \frac{x-6}{x^2+x-6}\,dx = \int \frac{9}{5}\frac{1}{x+3} - \frac{4}{5}\frac{1}{x-2}\,dx$$

$$= \frac{9}{5}\int \frac{1}{x+3}\,dx - \frac{4}{5}\int \frac{1}{x-2}\,dx$$

$$= \frac{9}{5}\ln(x+3) - \frac{4}{5}\ln(x-2) + C$$

Ex. $\int \frac{25}{x^3-1}\,dx = 25\int \frac{1}{(x-1)(x^2+x+1)}\,dx$

$x^3-1 = (x-1)(x^2+x+1)$

$$\begin{array}{r|rrrr} & x^3 & & & \\ 1 & 1 & 0 & 0 & -1 \\ & \downarrow & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$\underbrace{1 \quad 1 \quad 1}_{x^2}$

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$
$$1 = Ax^2+Ax+A + Bx^2 - Bx + Cx - C$$
$$1 = (A+B)x^2 + (A-B+C)x + (A-C)$$

$$\left.\begin{array}{ll} x^2: & 0 = A+B \\ x: & 0 = A-B+C \\ \#: & 1 = A-C \end{array}\right\} \quad A = -B$$

$$\left.\begin{array}{r} 0 = -2B+C \\ +\ 1 = -B-C \end{array}\right\} \quad C = 2B$$

$$1 = -3B$$

$$B = -1/3 \qquad A = +1/3$$
$$C = -2/3$$

So, $$\frac{1}{x^3-1} = \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{x+2}{x^2+x+1}$$

$$\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\underbrace{\int \frac{1}{x-1}\,dx}_{I_1} - \frac{1}{3}\underbrace{\int \frac{x+2}{x^2+x+1}\,dx}_{I_2}$$

$$I_1 = \ln(x-1)$$

$$I_2 = \frac{1}{2}\int \frac{2(x+2)}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2(x+1/2+3/2)}{x^2+x+1}\,dx$$

$u = x^2+x+1$

$du = (2x+1)\,dx$

$= 2(x+1/2)\,dx$

$$= \frac{1}{2}\int \frac{2(x+1/2)\,dx}{x^2+x+1} + \frac{1}{2}\int \frac{2(3/2)}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$\left(x^2+x+\tfrac{1}{4}\right)+1-\tfrac{1}{4} = \underbrace{(x+\tfrac{1}{2})^2}_{u^2} \underbrace{+\ \tfrac{3}{4}}_{+\ a^2} \qquad a=\frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}\int \frac{1}{(\underbrace{x+\tfrac{1}{2}}_{})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,d\cancel{x} = \frac{3}{2}\int \frac{1}{u^2+a^2}\,du = \underbrace{\frac{3}{2}\,\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C}_{}$$

$u = x+\tfrac{1}{2}$

$du = dx$

$$= \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\cdot\arctan\left(\frac{x+\tfrac{1}{2}}{\sqrt{3}/2}\right)+C$$

$$= \sqrt{3}\,\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

FINALLY,

$$\boxed{25\int\frac{1}{x^3-1}\,dx = \frac{25}{3}\ln(x-1) - \frac{25}{6}\ln(x^2+x+1) - \frac{25\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C}$$