M243 – Concept Check (§7.1–7.4) 9/17/18

Compute the antiderivatives:

1. $\int x^3 e^{3x}\,dx$

2. $\int e^{2x}\cos(\tfrac{1}{2}x)\,dx$

3. $\int \sin^3 x \cos^4 x\,dx$

4. $\int \tan x \sec^3 x\,dx$

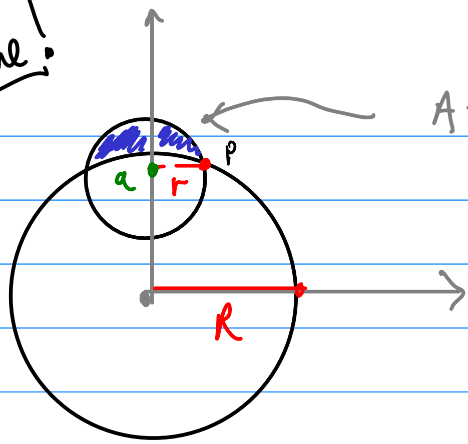
5. $\int \frac{x^3}{\sqrt{x^2+16}}\,dx$

6. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

7. $\int \frac{10}{(x-1)(x^2+9)}\,dx$

8. $\int \frac{x^4-2x^3+x^2+6x-5}{x^2-2x+1}\,dx$

7.3. last: **Lune!**



$$A = 2\int_0^r (\text{upper} - \text{lower})\,dx$$

Circle: $(x-h)^2+(y-k)^2=\rho^2$

small: $x^2+(y-a)^2=r^2$

big: $x^2+y^2=R^2$

P(r, a) lies on the larger circle, so, $r^2+a^2=R^2$, or $a=\sqrt{R^2-r^2}$

small: $y = a+\sqrt{r^2-x^2}$ upper

big: $y=\sqrt{R^2-x^2}$ lower

Find a in terms of the others.

$$A = 2\int_0^r \sqrt{R^2-r^2}+\sqrt{r^2-x^2}-\sqrt{R^2-x^2}\,dx$$

$$= 2\int_0^r \sqrt{R^2-r^2}\,dx + 2\int_0^r \sqrt{r^2-x^2}\,dx - 2\int_0^r \sqrt{R^2-x^2}\,dx$$

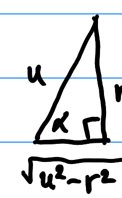
Take $\int_0^r \sqrt{u^2-x^2}\,dx = \int_0^\alpha \sqrt{u^2-u^2\sin^2\theta}\;u\cos\theta\,d\theta = u^2\int_0^\alpha \cos^2\theta\,d\theta = \frac{u^2}{2}\int_0^\alpha 1+\cos(2\theta)\,d\theta$

$x = u\sin\theta \rightarrow \sin\theta = \frac{x}{u} \rightarrow \theta = \arcsin\left(\frac{x}{u}\right)$

$dx = u\cos\theta\,d\theta$

$\theta(r) = \arcsin\left(\frac{r}{u}\right) = \alpha$

$\theta(0) = \arcsin(0) = 0$



$$= \frac{u^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{\theta=0}^{\alpha} = \frac{u^2}{2}\left(\alpha+\frac{1}{2}\sin(2\alpha) - 0 - 0\right) = \frac{u^2}{2}\left(\alpha + \sin\alpha\cos\alpha\right)$$

$$= \frac{u^2}{2}\left(\arcsin\left(\frac{r}{u}\right) + \frac{r}{u}\cdot\frac{\sqrt{u^2-r^2}}{u}\right) = \frac{u^2}{2}\arcsin\left(\frac{r}{u}\right) + \frac{r\sqrt{u^2-r^2}}{2}$$

So, $A = \underline{2r\sqrt{R^2-r^2}} + \underbrace{r^2\arcsin(1)}_{\pi/2} + \underbrace{\cancel{r(r^2-r^2)^{1/2}}}_{0} - R^2\arcsin\left(\frac{r}{R}\right) - \underline{r\sqrt{R^2-r^2}}$

$$\boxed{\text{Area of lune} = \frac{\pi r^2}{2} + r\sqrt{R^2-r^2} - R^2\arcsin\left(\frac{r}{R}\right)}$$

1. $\int x^3 e^{3x}\,dx$

	u	dv
$+$	x^3	e^{3x}
$-$	$3x^2$	$\frac{1}{3}e^{3x}$
$+$	$6x$	$\frac{1}{9}e^{3x}$
$-$	6	$\frac{1}{27}e^{3x}$
	0	$\frac{1}{81}e^{3x}$

$$\boxed{\int x^3 e^{3x}\,dx = \left(\frac{1}{3}x^3 - \frac{1}{3}x^2 + \frac{2}{9}x - \frac{2}{27}\right)e^{3x} + C}$$

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2. $\int e^{2x}\cos\left(\frac{1}{2}x\right)dx = 2e^{2x}\sin\left(\frac{1}{2}x\right) - \int 4e^{2x}\sin\left(\frac{1}{2}x\right)dx$

$u = e^{2x}$ $\quad$ $dv = \cos(\frac{1}{2}x)\,dx$ $\quad$ $u = e^{2x}$ $\quad$ $dv = \sin(\frac{1}{2}x)\,dx$

$du = 2e^{2x}dx$ $\quad$ $v = 2\sin(\frac{1}{2}x)$ $\quad$ $du = 2e^{2x}dx$ $\quad$ $v = -2\cos(\frac{1}{2}x)\,dx$

So,

$$\int e^{2x}\cos\left(\frac{1}{2}x\right)dx = 2e^{2x}\sin\left(\frac{1}{2}x\right) + 8e^{2x}\cos\left(\frac{1}{2}x\right) - 16\int e^{2x}\cos\left(\frac{1}{2}x\right)dx$$

$$17\int e^{2x}\cos\left(\frac{1}{2}x\right)dx = 2e^{2x}\sin\left(\frac{1}{2}x\right) + 8e^{2x}\cos\left(\frac{1}{2}x\right) + C$$

or

$$\boxed{\int e^{2x}\cos\left(\frac{1}{2}x\right)dx = \frac{2}{17}e^{2x}\sin\left(\frac{1}{2}x\right) + \frac{8}{17}e^{2x}\cos\left(\frac{1}{2}x\right) + C}$$

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3. $\int \sin^3 x\cos^4 x\,dx = \int \sin^2 x\cos^4 x\ \sin x\,dx = -\int (1-\cos^2 x)\cos^4 x\ {-\sin x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\int (1-u^2)u^4\,du = \int u^6 - u^4\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$\boxed{\int \sin^3 x\cos^4 x\,dx = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C}$$

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4. $\int \tan x \sec^3 x\,dx = \int \sec^2 x \quad \sec x \tan x\,dx \Rightarrow \int u^2\,du = \frac{1}{3}u^3 + C$

$u = \sec x$

$du = \sec x \tan x\,dx$

$$\boxed{\int \tan x \sec^3 x\,dx = \frac{1}{3}\sec^3 x + C}$$

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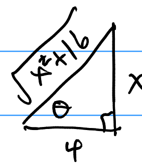
5. $\int \frac{x^3}{\sqrt{x^2+16}}\,dx$

$x = 4\tan\theta \rightarrow \tan\theta = \frac{x}{4} \rightarrow$  $\rightarrow \sec\theta = \frac{\sqrt{x^2+16}}{4}$

$dx = 4\sec^2\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{x^2+16}}\,dx = \int \frac{(4\tan\theta)^3}{\sqrt{(4\tan\theta)^2+16}}\,4\cdot\sec^2\theta\,d\theta = \int \frac{4^{\cancel{4}\,3}\tan^3\theta\sec^{\cancel{2}}\theta}{\cancel{4}\,\cancel{\sec\theta}}\,d\theta$$

$$= 64\int \tan^3\theta \sec\theta\,d\theta = 64\int \tan^2\theta \quad \sec\theta\tan\theta\,d\theta = 64\int (\sec^2\theta - 1) \quad \sec\theta\tan\theta\,d\theta$$

$$= 64\int u^2 - 1\,du = 64\left(\frac{1}{3}u^3 - u + C\right)$$

$$= 64\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C = 64\left(\frac{1}{3}\frac{\sqrt{x^2+16}^{\,3}}{64} - \frac{\sqrt{x^2+16}}{4}\right) + C$$

So, $$\boxed{\int \frac{x^3}{\sqrt{x^2+16}}\,dx = \frac{1}{3}(x^2+16)\sqrt{x^2+16} - 16\sqrt{x^2+16} + C}$$

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6. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^2}{\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\,d\theta = 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int 1 - \cos(2\theta)\,d\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C = \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$$

$\sin\theta = \frac{x}{3} \rightarrow \theta = \arcsin\left(\frac{x}{3}\right) \rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{9}\right) + C$$

So, $$\boxed{\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C}$$

$$7.\quad \int \frac{10}{(x-1)(x^2+9)}\,dx$$

PFD: $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9} = \dfrac{1}{x-1} - \dfrac{x+1}{x^2+9}$

$\Rightarrow \quad 10 = A(x^2+9) + (Bx+C)(x-1)$

$x=1: \quad 10 = 10A \quad \Rightarrow \quad A = 1.$

$\Rightarrow \quad 10 = x^2+9 + Bx^2 + (C-B)x - C$

$10 = (B+1)x^2 + (C-B)x + (9-C)$

$0 = B+1 \quad \Rightarrow \quad B = -1$

$0 = C-B \quad \Rightarrow \quad C = -1$

$10 = 9-(-1) = 10 \quad \checkmark$

So, $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{x+1}{x^2+9}\,dx = \int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$

and
$$\boxed{\int \frac{10}{(x-1)(x^2+9)}\,dx = \ln(x-1) - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C}$$

$$8.\quad \int \frac{x^4-2x^3+x^2+6x-5}{x^2-2x+1}\,dx$$

PFD: Divide first:

$$\begin{array}{r l}
 & x^2 + \dfrac{6x-5}{x^2-2x+1} \\
x^2-2x+1 \,\big) & x^4-2x^3+x^2+6x-5 \\
 & -(x^4-2x^3+x^2) \\
\hline
 & 0 + 6x - 5
\end{array}$$

$= \displaystyle\int x^2 + \frac{6x-5}{(x-1)^2}\,dx$

OR, since $x^2-2x+1 = (x-1)^2$:

$$\begin{array}{c|ccccc}
1 & 1 & -2 & 1 & 6 & -5 \\
 & & 1 & -1 & 0 & 6 \\
\hline
1 & 1 & -1 & 0 & 6 & \boxed{1} \\
 & & 1 & 0 & 0 & \\
\hline
 & 1 & 0 & 0 & \boxed{6} &
\end{array}$$

So, $\dfrac{x^4-2x^3+x^2+6x-5}{x^2-2x+1} = x^2 + \dfrac{6 + \frac{1}{x-1}}{x-1}$

$= x^2 + \dfrac{6x-5}{(x-1)^2}$

$= \displaystyle\int x^2\,dx + 6\int \frac{1}{x-1}\,dx + 1\int \frac{1}{(x-1)^2}\,dx$

$$\boxed{= \frac{1}{3}x^3 + 6\ln(x-1) - \frac{1}{x-1} + C}$$

Now, $\dfrac{6x-5}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} = \dfrac{6}{x-1} + \dfrac{1}{(x-1)^2}$

NOT a coincidence!

$6x-5 = A(x-1) + B = Ax + (B-A)$

$A = 6$

$-5 = B-6 \quad \Rightarrow \quad B = 1$