M243 — §7.5/7.6 — Integration using tables/CAS

9/18/18

Compute the integrals.

1. $\int_0^{\pi} t\cos^2 t\,dt$

2. $\int_{-1}^{1} \frac{e^{\arctan(y)}}{1+y^2}\,dy$

3. $\int_0^1 (1+\sqrt{x})^8\,dx$

4. $\int \theta\tan^2\theta\,d\theta$

5. $\int \frac{1}{x\sqrt{4x+1}}\,dx$

6. $\int \frac{e^{2x}}{1+e^x}\,dx$

$$\frac{1}{2}\int_2^6 \frac{2(x+2-2)}{x^2+4x+20}\,dx = \underbrace{\frac{1}{2}\int_2^6 \frac{2(x+2)\,dx}{x^2+4x+20}} - 2\int_2^6 \frac{1}{x^2+4x+20}\,dx$$

$u = x^2+4x+20$

$du = 2x+4\,dx = 2(x+2)\,dx$

$$\frac{1}{2}\int_{x=2}^{6} \frac{1}{u}\,du - 2\int_2^6 \frac{1}{x^2+4x+4+20-4}\,dx$$

$u(6) = 36+24+20 = 80$

$u(2) = 4+16+20 = 40$

$$= \frac{1}{2}\int_{x=2}^{6} \frac{1}{u}\,du - 2\int_2^6 \frac{1}{(x+2)^2+4^2}\,dx$$

$$= \frac{1}{2}\ln 80 - \frac{1}{2}\ln 40 - 2\cdot\frac{1}{4}\arctan\left(\frac{x+2}{4}\right)\Bigg|_{x=2}^{6}$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\arctan(2) + \frac{1}{2}\arctan(1)$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\arctan(2) + \pi/8$$

1. $\int_0^{\pi} t\cos^2 t\,dt$

$u = t \qquad dv = \cos^2 t\,dt = \frac{1}{2}(1+\cos(2t))\,dt$

$du = dt \qquad v = \frac{1}{2}t + \frac{1}{4}\sin(2t)$

$$= \frac{1}{2}t^2 + \frac{1}{4}t\sin(2t)\Big|_{t=0}^{\pi} - \int_0^{\pi} \frac{1}{2}t + \frac{1}{4}\sin(2t)\,dt$$

$$= \frac{1}{2}t^2 + \frac{1}{4}t\sin(2t) - t^2 + \frac{1}{8}\cos(2t)\Big|_0^{\pi} = -\frac{\pi^2}{2}$$

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2. $$\int_{-1}^{1} \frac{e^{\arctan(y)}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = e^u\Big|_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4} = 2\sinh(\pi/4)$$

$u = \arctan(y) \qquad u(1) = \arctan(1) = \pi/4$

$du = \frac{1}{1+y^2}\,dy \qquad u(-1) = \arctan(-1) = -\pi/4$

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3. $$\int_0^1 (1+\sqrt{x})^8\,dx = \int_0^1 \frac{2\sqrt{x}\,(1+\sqrt{x})^8}{2\sqrt{x}}\,dx = 2\int_1^2 (u-1)u^8\,du = 2\int_1^2 u^9 - u^8\,du$$

$u = 1+\sqrt{x} \qquad u(1) = 2 \qquad \sqrt{x} = u-1$

$du = \frac{1}{2\sqrt{x}}\,dx \qquad u(0) = 1$

$$= 2\left(\frac{u^{10}}{10} - \frac{u^9}{9}\right)\Big|_1^2 = 2\left(\frac{2^{10}}{10} - \frac{2^9}{9} - \frac{1}{10} + \frac{1}{9}\right)$$

$$= 2\left(\frac{1023}{10} - \frac{511}{9}\right) = \frac{4097}{45} = 91.0\overline{4}$$

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4. $$\int \theta\tan^2\theta\,d\theta = \theta\tan\theta - \theta^2 - \int \tan\theta - \theta\,d\theta = \theta\tan\theta - \frac{1}{2}\theta^2 - \int \tan\theta\,d\theta$$

$u = \theta \qquad dv = \tan^2\theta\,d\theta = (\sec^2\theta - 1)\,d\theta$

$du = d\theta \qquad v = \tan\theta - \theta$

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta = -\int \frac{1}{u}\,du = -\ln|u|$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= -\ln|\cos\theta|$$
$$= \ln|\cos\theta|^{-1}$$
$$= \ln|\sec\theta|$$

So, $\boxed{\int \theta\tan^2\theta\,d\theta = \theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C}$

5. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx = \int \frac{1}{(u-1)\sqrt{u}}\,du = \int \frac{2}{(\sqrt{u}^2-1)\,2\sqrt{u}}\,du = \int \frac{2}{v^2-1}\,dv \overset{\text{Now}}{=\!=} \begin{cases} \text{H.Trig} \\ \text{or} \\ \text{P.F.D.} \end{cases}$

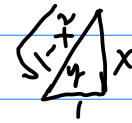
$u = 4x+1 \qquad x = \frac{1}{4}(u-1)$

$du = 4\,dx$

$v = \sqrt{u}$

$dv = \frac{1}{2\sqrt{u}}\,du$

H.Trig:



$y = \operatorname{arctanh}(x) \Rightarrow \tanh(y) = x$

$\Rightarrow \operatorname{sech}^2(y)\,\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \cosh^2 x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$

P.F.D.: $\frac{2}{v^2-1} = \frac{A}{v-1} + \frac{B}{v+1} = \frac{1}{v-1} - \frac{1}{v+1}$

$\Rightarrow 2 = A(v+1) + B(v-1)$

$v=1: \quad 2 = 2A \Rightarrow A = 1$

$v=-1: \quad 2 = -2B \Rightarrow B = -1$

$= \displaystyle\int \frac{2}{v^2-1}\,dv = \begin{cases} -2\operatorname{arctanh}(v) + C \\ \text{or} \\ \ln(v-1) - \ln(v+1) = \ln\left(\frac{v-1}{v+1}\right) + C \end{cases}$

Plug back in $v = \sqrt{4x+1}$ to get

$$\int \frac{1}{x\sqrt{4x+1}}\,dx = -2\operatorname{arctanh}\left(\sqrt{4x+1}\right) + C \overset{\text{OR}}{=} \ln\left(\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right) + C$$

6. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{e^x}{1+e^x}\,e^x\,dx = \int \frac{u}{1+u}\,du = \int \frac{u+1-1}{u+1}\,du = \int 1 - \frac{1}{u+1}\,du$

$u = e^x$

$du = e^x\,dx$

$= u - \ln(u+1) + C = e^x - \ln(e^x+1) + C$