M243 9/19

§7.7 – Approximate Integrals

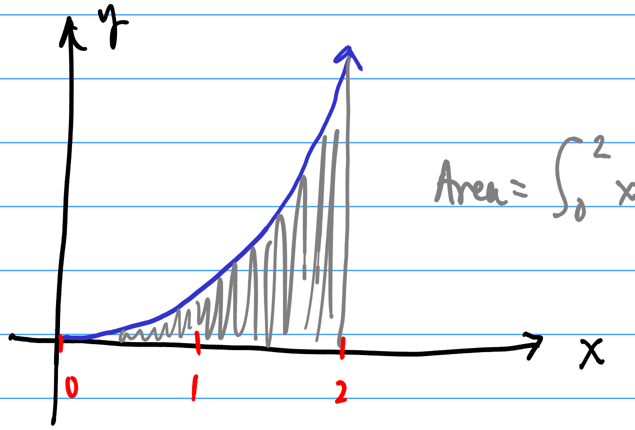
FTC: If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

e.g., $\int_0^1 e^{x^2}\,dx$ ←

$u = x^2$
$du = 2x\,dx$] X — we can't do it via antiderivative methods.

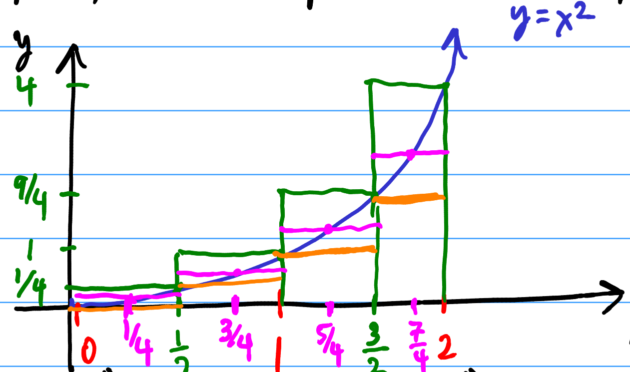
Ex. $\int_0^2 x^2\,dx$



Area $= \int_0^2 x^2\,dx$

Review Right endpoint, left endpoint, and Midpoint rules for $n=4$.

$\int_a^b f(x)\,dx$



R_4 M_4 L_4

$\overset{\shortparallel}{x_0} = a$ $\quad \frac{1}{2} = x_1 \quad \overset{\shortparallel}{x_2} \quad \overset{\shortparallel}{x_3} \quad x_4 = b$

base $= \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

heights $= f(x_i^*)$

Right Endpoint Rule: choose x_i^* to be the right end points of each interval.

$x_i^* = a + i\Delta x$ ← Right endpoints!

i	x_i^*	$f(x_i^*) = (x_i^*)^2$	$f(x_i^*)\cdot\Delta x$
1	1/2	1/4	1/8
2	1	1	1/2
3	3/2	9/4	9/8
4	2	4	2
			$\boxed{\text{SUM}}$

$R_4 = \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8} = \frac{30}{8} = \frac{15}{4}$

$= 3.75$

→ $R_4 = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$

Left endpoint Rule:

$\Delta x = \frac{b-a}{n} = 1/2$

$a = x_0$

$x_i^* = a + (i-1)\Delta x$

i	x_i^*	$f(x_i^*) = (x_i^*)^2$	$f(x_i^*)\Delta x$
1	0	0	0
2	1/2	1/4	1/8
3	1	1	1/2
4	3/2	9/4	9/8
			$\boxed{\text{Sum}}$

$L_4 = 1/8 + 4/8 + 9/8 = 14/8 = 7/4 = 1.75$

Midpoint Rule:

$\Delta x = \frac{b-a}{n} = \frac{1}{2}$

$x_i^* = (a + \frac{1}{2}\Delta x) + (i-1)\Delta x = a + (i - \frac{1}{2})\Delta x$

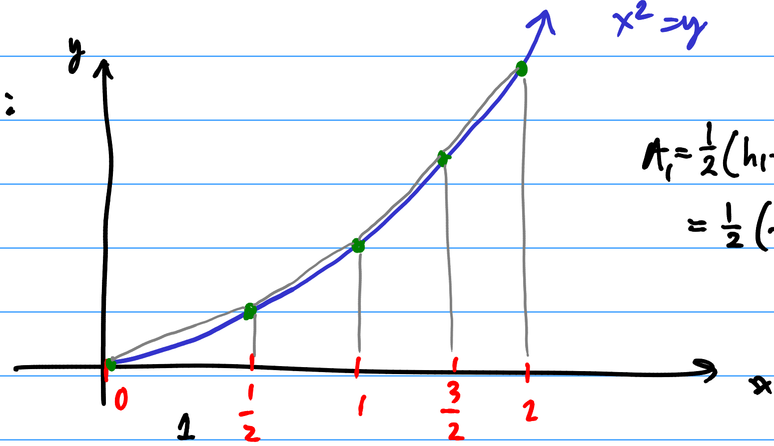
i	x_i^*	$f(x_i^*)$	$f(x_i^*)\Delta x$
1	1/4	1/16	1/32
2	3/4	9/16	9/32
3	5/4	25/16	25/32
4	7/4	49/16	49/32
			$\boxed{\text{Sum}}$

$M_4 = \frac{1}{32}(1 + 9 + 25 + 49) = \frac{84}{32} = \frac{21}{8}$

$= 2.625$

Ex. $\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3} = 2.\overline{66}$ exact answer.

Trapezoidal Rule:



$A_i = \frac{1}{2}(h_1+h_2)\cdot b$
$= \frac{1}{2}\left(f(x_{i-1}^*)+f(x_i^*)\right)\cdot \Delta x$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}\left(f(x_{i-1}^*)+f(x_i^*)\right)\Delta x$$

$$T_n = \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right)$$

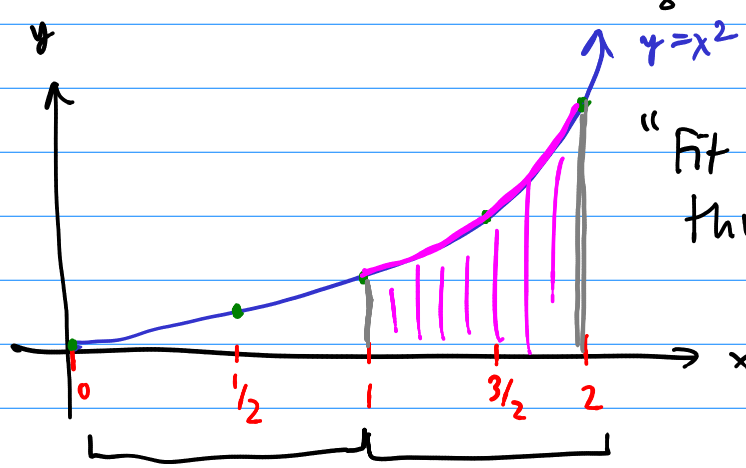
i	x_i	$f(x_i)=x_i^2$		Trap	(Trap) $\frac{\Delta x}{2}$
0	0	0	×1	0	0
1	1/2	1/4	2	1/2	1/8
2	1	1	2	2	1/2
3	3/2	9/4	2	9/2	9/8
4	2	4	1	4	1

$\boxed{\text{Sum}}$

$T_4 = 0 + \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 1 =$
$\frac{1+4+9+8}{8} = \frac{11}{4} = 2.75$

Simpson's Rule

$n =$ even



"Fit a parabola through 3 pts"

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

n even

i	x_i	$f(x_i) = x_i^2 \rightarrow$		Simp	$(\text{Simp})\frac{\Delta x}{3}$
0	0	0	$\times 1$	0	0
1	$1/2$	$1/4$	4	1	$1/6$
2	1	1	2	2	$2/6$
3	$3/2$	$9/4$	4	9	$9/6$
4	2	4	1	4	$4/6$

$S_4 = \boxed{\text{Sum}}$

$S_4 = \frac{1}{6}(1+2+9+4) = \frac{16}{6} = \frac{8}{3} = 2.\overline{6}$ EXACT!

Simpson's Rule will always give the exact answer for quadratic functions.