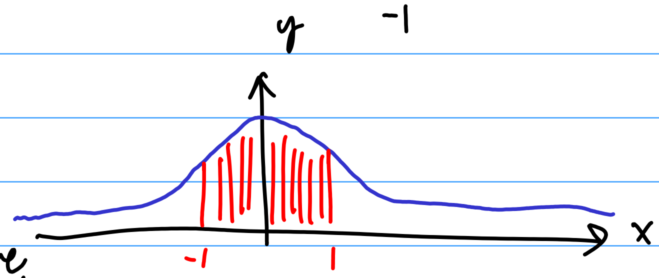
M243 — 9/20/18

Ex. Approximate the integral $\int_{-1}^{1} e^{-x^2}\,dx \approx 2.925$ by $M_{100}, T_{100}, S_{100}$

$y = e^{-x^2}$



[y does **not** have a simple antiderivative.

Instead we'll use M_n, T_n, S_n

$$M_n = \Delta x\left[f\left(\tfrac{x_0+x_1}{2}\right) + f\left(\tfrac{x_1+x_2}{2}\right) + \cdots + f\left(\tfrac{x_{n-1}+x_n}{2}\right)\right]$$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$ n is even

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Q. How accurate are these methods?

We can bound the error!

Error: $E_{M_n} = \int_a^b f(x)\,dx - M_n$ b/c $\int_a^b f(x)\,dx = M_n + E_{M_n}$

Theorem: Suppose $f''(x) \leq K$ on $[a,b]$, then

$$\boxed{|E_{M_n}| \leq \frac{K(b-a)^3}{24n^2}} \quad \text{and} \quad \boxed{|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2}}$$

Thm. Suppose $f^{(4)}(x) \le K$ for all $[a,b]$. Then

$$\boxed{|E_{S_n}| \le \frac{K(b-a)^5}{180\,n^4}} \leftarrow$$

Ex. Simpson's Rule gives the exact answer for any cubic function.

$$f(x) = ax^3 + bx^2 + cx + d$$
$$f'(x) = 3ax^2 + 2bx + c$$
$$f''(x) = 6ax + 2b$$
$$f'''(x) = 6a$$
$$f^{(4)}(x) = 0 \Rightarrow \text{ we can set } K = 0, \text{ and}$$

$$|E_{S_n}| \le \frac{K(b-a)^5}{180\,n^4} = \frac{0\,(b-a)^5}{180\,n^4} = 0.$$

so $\int_A^B (ax^3 + bx^2 + cx + d)\,dx = S_2$.

Ex. How large should we choose n to ensure that Simpson's rule has error less than or equal to $\underbrace{0.00001}$ when approximating

$$\int_0^1 e^{x^2}\,dx$$

$tol = \frac{1}{100{,}000}$ want: $|E_{S_n}| \le \frac{1}{100{,}000}$

$$\uparrow$$

$$\le \frac{K(b-a)^5}{180\,n^4} = \frac{K}{180\,n^4}$$

$$\frac{K}{180\,n^4} \le \frac{1}{100{,}000} \Rightarrow \frac{180\,n^4}{K} \ge 100{,}000$$

solve for n.

So, $n \geq \sqrt[4]{\dfrac{100{,}000\, k}{180}}$

$|f^{(4)}(x)| \leq k$ on $[0,1]$

$f(x) = e^{x^2}$

$f'(x) = 2x\, e^{x^2}$

$f''(x) = (2 + 4x^2)\, e^{x^2}$

$f'''(x) = (8x + 4x + 8x^3)\, e^{x^2} = (12x + 8x^3)\, e^{x^2}$

$f^{(4)}(x) = (12 + 24x^2 + 24x^2 + 16x^4)\, e^{x^2} = \boxed{(12 + 48x^2 + 16x^4)\, e^{x^2}}$

$[0,1]$

$f^{(4)}(0) = 12$

$f^{(4)}(1) = 76e$ RE. Also check the CNs inside $(0,1)$.

$K = 76e$

$n \geq \sqrt[4]{\dfrac{76e(100{,}000)}{180}} \approx 18.4$

Since we're doing Simpson's Rule, n must be even, so

choose $\boxed{n = 20}$