M242 9/21/18

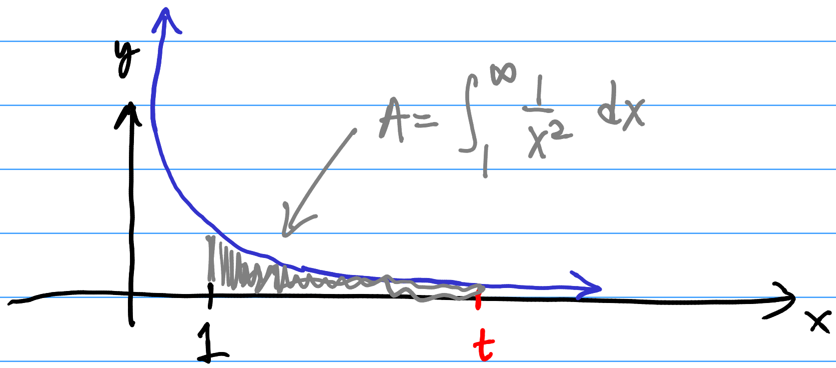
§7.8 - Improper Integrals

"Something is tending to ∞"

Case I. $x \to \infty$

Motivating Example:

$f(x) = \frac{1}{x^2}$



Idea: for any $t > 1$, we can compute $\int_1^t \frac{1}{x^2}\,dx$

$$\int_1^t \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^t = \frac{-1}{t} - \left(\frac{-1}{1}\right) = 1 - \frac{1}{t}$$

$$x^{-2} \longrightarrow \frac{x^{-1}}{-1}$$

Then, use a limit to "push" t to ∞.

$$A = \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 - 0 = 1$$

Defn. If $\int_a^t f(x)\,dx$ exists for all $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

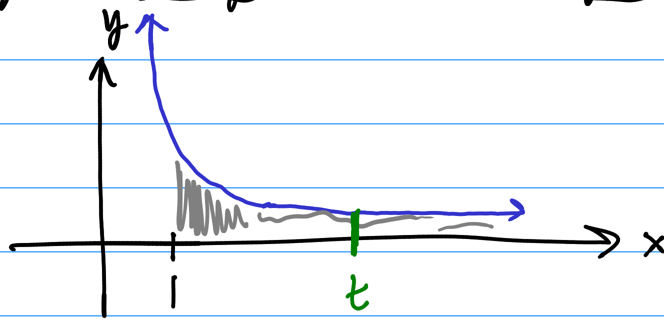
If $\int_t^b f(x)\,dx$ exists for all $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Providing the limits exist.

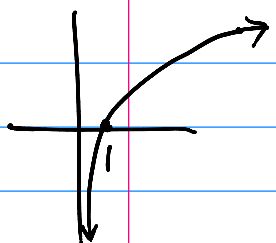
Def'n. If the limits exist, then we say that the integrals are convergent.
If not, the integrals diverge (or are divergent).

Ex. $\int_1^\infty \frac{1}{x}\,dx$



$y = \ln_e 1 \rightarrow e^y = 1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \underbrace{\int_1^t \frac{1}{x}\,dx} = \lim_{t\to\infty}\left[\ln x \Big|_1^t\right] = \lim_{t\to\infty}\left[\ln t - \ln 1\right]$$



$$= \lim_{t\to\infty} \ln t = \infty$$ so this integral DIVERGES.

Thm. (P-test) $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if and only if $p>1$.
where $a>0$.

Proof. $\int_a^\infty \frac{1}{x^p}\,dx$ w/ $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{1-p}\Big|_1^t\right]$$

x^{-p}

$$= \lim_{t\to\infty}\left[\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right] = \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p}-1\right)$$

$$= \frac{1}{1-p}\left[\left(\lim_{t\to\infty} t^{1-p}\right) - 1\right]$$

$$\lim_{t\to\infty} t^{1-p} = \lim_{t\to\infty} \frac{t}{t^p} = \lim_{t\to\infty} \frac{1}{t^{p-1}} = \begin{cases} 0 & p>1 \leftarrow \\ \infty & p<1 \leftarrow \end{cases}$$

So the integral is convergent if $p>1$
and divergent if $p \le 1$.

Ex. $\frac{1}{3}\int_1^\infty \frac{3x^2}{(1+x^3)^4}\,dx$ Converge or Diverge?

$$\left.\begin{aligned} u &= 1+x^3 \\ du &= 3x^2\,dx \end{aligned}\right\} \quad \frac{1}{3}\int_2^\infty \frac{1}{u^4}\,du$$

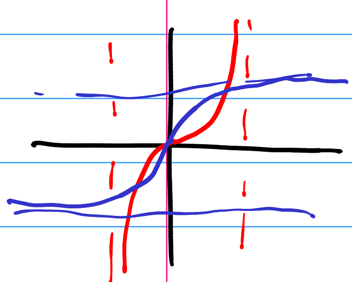
Convergent by p-T!

Ex. $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{u\to\infty} \int_0^u \frac{1}{1+x^2}\,dx$$

$$\lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\left[\arctan(x)\Big|_t^0\right] = \lim_{t\to\infty}\left(\arctan(0) - \arctan(t)\right)$$



$$= 0 - \lim_{t\to\infty} \arctan(t) = 0 - \left(-\pi/2\right) = \pi/2$$

other half: $$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{u\to\infty}\left(\arctan(u) - \arctan 0\right)$$

$$= \pi/2 - 0 = \pi/2$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \pi/2 + \pi/2$$

$$= \pi$$

$f(x) = \dfrac{1}{1+x^2}$ "~~witch~~ of Agnesi"

curve