M243 9/24/18

§7.8 - Improper Integrals

Type I: $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

Ex. $\displaystyle\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \int_t^0 xe^x\,dx = \lim_{t\to-\infty}\left(xe^x - \int e^x\,dx\right)\Big|_t^0$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$\displaystyle = \lim_{t\to-\infty}\left(xe^x - e^x\right)\Big|_t^0$

$\displaystyle = \lim_{t\to-\infty}\left(\underbrace{0e^0}_{0} - \underbrace{e^0}_{1} - te^t + e^t\right)$

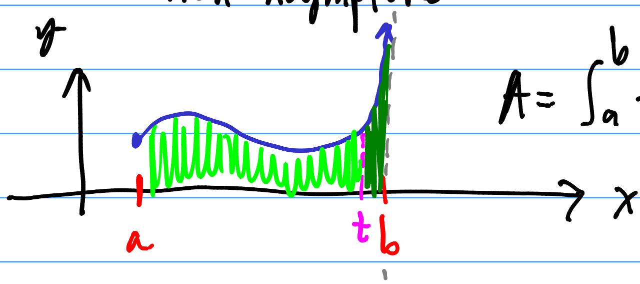
$\displaystyle = -1 - \lim_{t\to-\infty} te^t + \underbrace{\lim_{t\to-\infty} e^t}_{=0}$

$\displaystyle = -1 - \underbrace{\text{"}-\infty\cdot 0\text{"}}_{\text{more work}} = -1 - \lim_{t\to-\infty} \frac{t}{e^{-t}} \overset{\text{LH}}{=} -1 - \lim_{t\to-\infty}\frac{1}{-e^{-t}}$

$\displaystyle = -1 + \lim_{t\to-\infty} e^t = -1 + 0 = \boxed{-1}$ converges

---

Type II. $y \to \pm\infty$: Vertical Asymptotes!



$\displaystyle A = \int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

or

$\displaystyle \int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

Ex. $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ "Problem: $\frac{1}{\sqrt{2-2}}$ = undef."

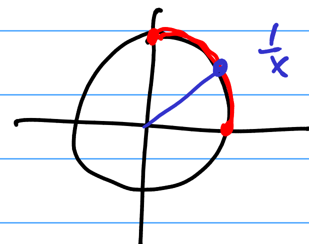
$$= \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 0^+}\int_t^3 u^{-1/2}\,du = 2u^{1/2}\Big|_t^3$$

$u = x-2 \rightarrow u(5) = 5-2 = 3$

$du = dx \quad u(2) = 2-2 = 0$

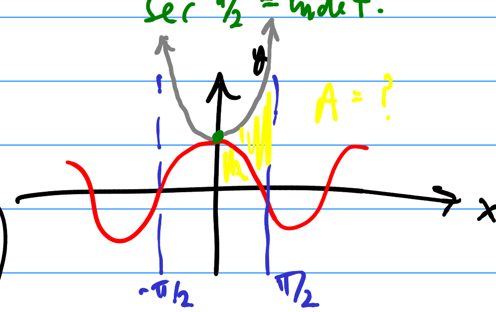
$$= \lim_{t\to 0^+} 2\sqrt{u}\Big|_t^3 = 2\sqrt{3} - \lim_{t\to 0^+} 2\sqrt{t} = 2\sqrt{3} - 0 = \boxed{2\sqrt{3}}$$

Ex. $\int_0^{\pi/2} \sec\theta\, d\theta$



$$= \lim_{t\to \pi/2^-}\int_0^t \sec\theta\, d\theta$$

sec π/2 = undef.



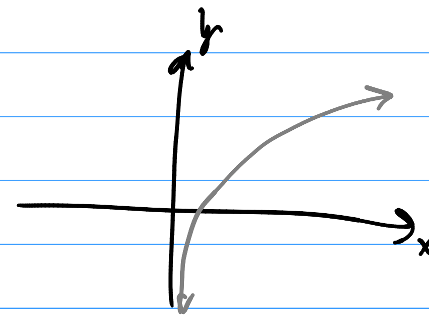
$$= \lim_{t\to \pi/2^-} \ln|\sec\theta + \tan\theta|\Big|_0^t$$

$$= \lim_{t\to \pi/2^-}\left(\ln|\sec t + \tan t| - \underbrace{\ln|\sec 0 + \tan 0|}_{=0}\right)$$

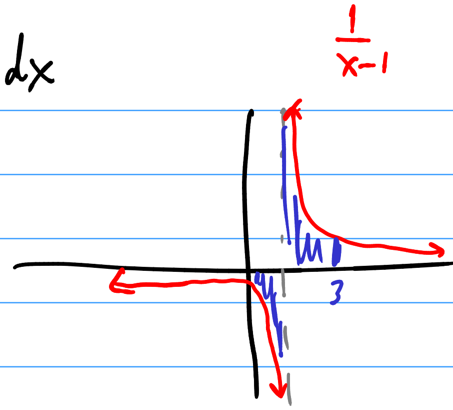
$$= \lim_{t\to \pi/2^-} \ln|\sec t + \tan t|$$

$$\underset{\text{cont.}}{=} \ln\left(\lim_{t\to \pi/2^-}\sec t + \lim_{t\to \pi/2^-}\tan t\right) = \ln|+\infty + \infty| = +\infty$$

So $\int_0^{\pi/2} \sec\theta\, d\theta$ is divergent.



Ex. $\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$



$= \int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$

$= \lim_{t\to 1^-} \left(\ln|x-1| \Big|_0^t \right)$

$= \lim_{t\to 1^-} \left(\ln|t-1| - \cancel{\ln|0-1|} \right)$ 0

$= \lim_{t\to 1^-} \ln|t-1| = \lim_{u\to 0^+} \ln u = -\infty$ = DIVERGES

$\int_0^3 \frac{1}{x-1}\,dx$ is divergent.

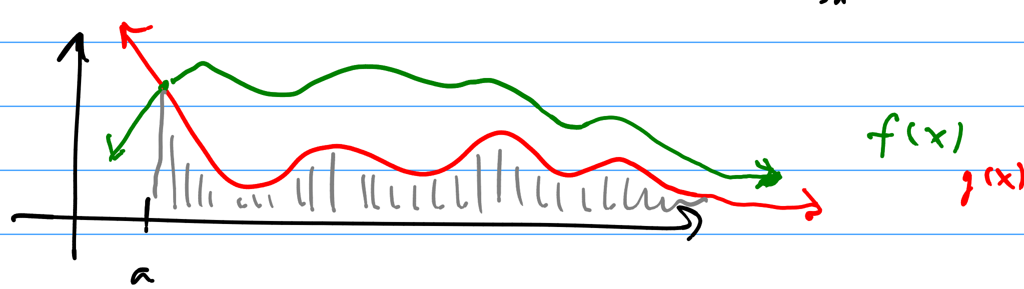
Thm. Comparison Thm.

Suppose f and g are continuous functions with:

$$f(x) \geq g(x) \geq 0 \quad \text{for} \quad x \geq a$$

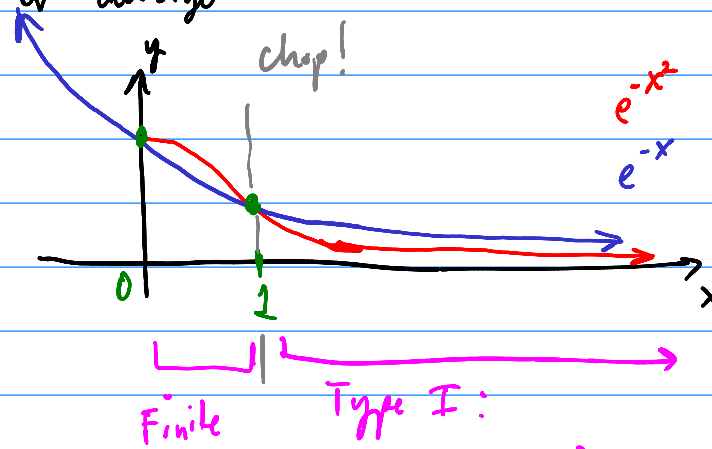
Then, 1. If $\int_a^\infty f(x)\,dx$ is convergent, then so is $\int_a^\infty g(x)\,dx$.

2. If $\int_a^\infty g(x)\,dx$ is divergent, then so is $\int_a^\infty f(x)\,dx$.



Ex. $\int_0^\infty e^{-x^2}\,dx$ converge or diverge

Compare to $\int_0^\infty e^{-x}\,dx$



$e^{-x} \geq e^{-x^2}$

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\#} + \underbrace{\int_1^\infty e^{-x^2}dx}_{?}$$

but $\int_1^\infty e^{-x^2}dx \leq \underbrace{\int_1^\infty e^{-x}dx}_{\text{compute!}}$

$$\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big]_1^t = \left(\cancel{-e^{-t}}^{\,0} + \underbrace{e^{-1}}\right) \xrightarrow{t\to\infty}$$

$$= \frac{1}{e} \quad : \quad \underline{\text{convergent}}.$$

we deduce by CT: $\underbrace{\int_0^\infty e^{-x^2}dx}_{=\frac{\sqrt{\pi}}{2}}$ is <u>convergent</u>.

Do the <u>HW</u>!