M243 9/26/18

7.4.10 Evaluate the integral.

$$\frac{1}{2}\int_7^8 \frac{\boxed{2x+4}-4}{x^2+4x+13}\,dx$$

$u = x^2+4x+13$

$du = (2x+4)\,dx$

$2(x+2)$

$$(x^2+4x+4)+(13-4)$$

$$= (x+2)^2+3^2$$

$$= \int_7^8 \frac{x}{(x+2)^2+3^2}\,dx$$

$u = x+2 \qquad x = u-2 \qquad u(8) = 8+2 = 10$

$du = dx \qquad u(7) = 9$

$$\int_9^{10} \frac{u-2}{u^2+3^2}\,du = \frac{1}{2}\int_9^{10}\frac{2u}{u^2+3^2}\,du - 2\int_9^{10}\frac{1}{u^2+3^2}\,du$$

$v = u^2+3^2 \qquad v(10) = 10^2+3^2 = 109$

$dv = 2u\,du \qquad v(9) = 9^2+3^2 = 90$

$$\frac{1}{2}\int_{90}^{109}\frac{1}{v}\,dv - 2\int_9^{10}\frac{1}{u^2+3^2}\,du$$

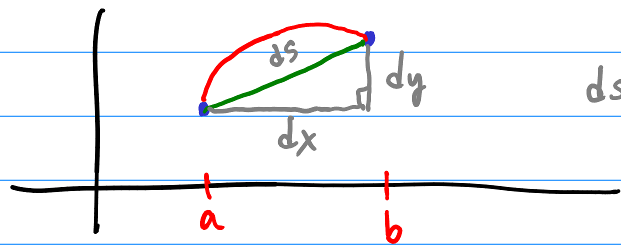
$$= \frac{1}{2}\ln|v|\Big|_{90}^{109} - \frac{2}{3}\arctan\left(\frac{u}{3}\right)\Big|_9^{10}$$

$$= \underbrace{\frac{1}{2}\ln(109)-\frac{1}{2}\ln(90)} - \frac{2}{3}\arctan\left(\frac{10}{3}\right) + \frac{2}{3}\arctan\left(\frac{9}{3}\right)$$

= compute ... =

$$= \ln\left(\sqrt{\frac{109}{90}}\right) - \frac{2}{3}\arctan\left(\frac{10}{3}\right) + \frac{2}{3}\arctan(3)$$

§ 8.1 Arc Length

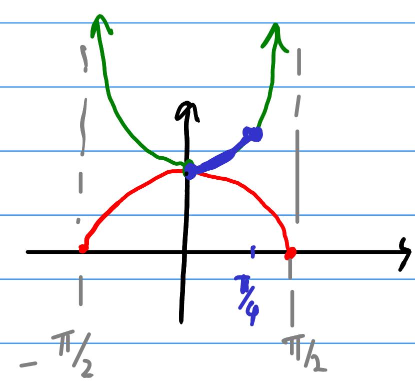


$ds = \sqrt{dx^2 + dy^2}$

$$l = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \underbrace{\int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

Ex. $y = \ln(\sec x) \qquad 0 \le x \le \pi/4$

$$l = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$y = \ln(\sec x)$

$$\frac{dy}{dx} = \frac{u'}{u} = \frac{\cancel{\sec x}\tan x}{\cancel{\sec x}} = \tan x$$

$\left(\frac{dy}{dx}\right)^2 = \tan^2 x \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\sec^2 x} = \sec x$$

$$l = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln\left(\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right) - \ln(\sec 0 - \tan 0)$$

$$= \ln(\sqrt{2} + 1) - \ln(1 - 0)$$

$$= \ln(\sqrt{2}+1) - \ln(1) = \boxed{\ln(\sqrt{2}+1)}$$

Ex. $y = \frac{x^3}{3} + \frac{1}{4x}$ $\qquad 1 \le x \le 2$ $\qquad \int \underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \, dx$

$$\frac{dy}{dx} = \frac{3x^2}{3} - \frac{1}{4x^2} = x^2 - \frac{1}{4x^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 - 1/2 + \frac{1}{16x^4}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = x^4 + 1/2 + \frac{1}{16x^4} \overset{?}{=} \underbrace{\left(x^2 + \frac{1}{4x^2}\right)^2} = x^4 + \frac{1}{2} + \frac{1}{16x^4} \checkmark$$

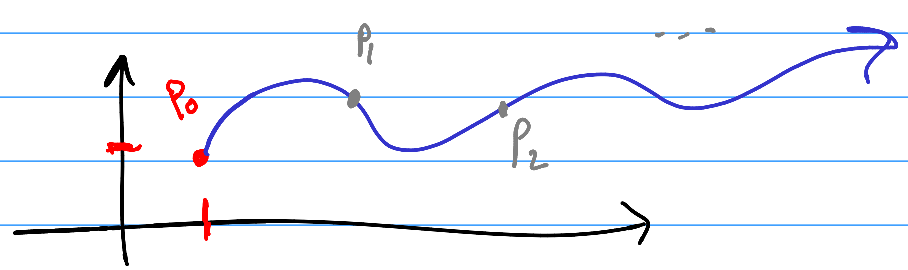
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^4 + 1/2 + \frac{1}{16x^4}}$$

$$= \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} = x^2 + \frac{1}{4x^2}$$

Now, $\int_1^2 x^2 + 1/4x^2 \, dx = \left.\frac{x^3}{3} - \frac{1}{4x}\right|_1^2 = \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$

$$= \frac{7}{3} + \frac{1}{8} = \boxed{\frac{59}{24}}$$

Q. what if we want to know the arc length of the same curve w/ different boundaries?



We can make an arc length function:

$$s(t) = \int_{x_0}^{t} \sqrt{1+(f'(x))^2} \, dx$$

$\underline{Ex}.$ $y = x^2 - \frac{1}{8}\ln x$ $\qquad x_0 = 1 \leftarrow$ start point.

Find $s(t) = \int_1^t \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = x^2 - \frac{1}{8}\ln x$

$\frac{dy}{dx} = 2x - \frac{1}{8x}$

$\left(\frac{dy}{dx}\right)^2 = 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$

$1+\left(\frac{dy}{dx}\right)^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$

$\sqrt{\left(2x+\frac{1}{8x}\right)^2} = \underbrace{2x + \frac{1}{8x}} \leftarrow$ T.B.I.

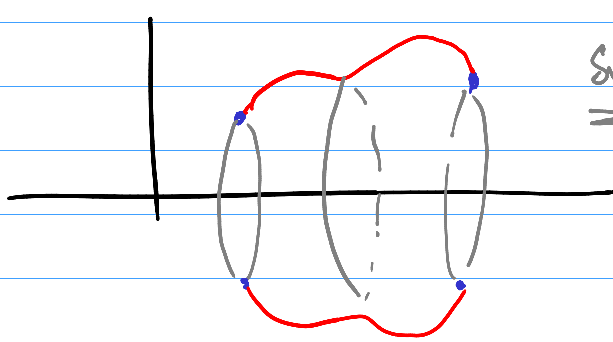
$s(t) = \int_1^t 2x + \frac{1}{8x}\,dx = x^2 + \frac{1}{8}\ln|x| \Big|_1^t, \qquad t \geq 1$

$s(t) = t^2 + \frac{1}{8}\ln t - 1^2 - \frac{1}{8}\ln 1$

$\boxed{s(t) = t^2 + \frac{1}{8}\ln t - 1}$

$\underline{Ex}.$ $s(e) = e^2 + \frac{1}{8}\ln e - 1 = e^2 + \frac{1}{8} - 1 = e^2 - \frac{7}{8}$

$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 9 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{1}{8}\ln 3$



Surface area ?