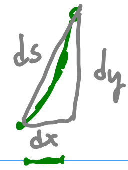
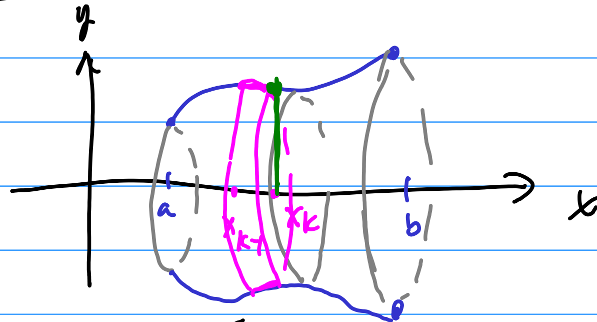
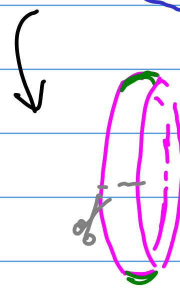
M243 9/27/18

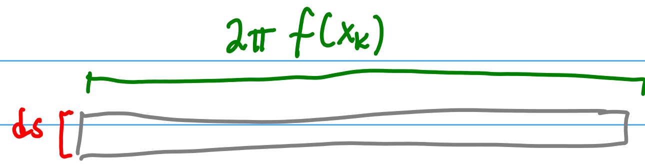




Q: Compute the surface area.



SA = ?



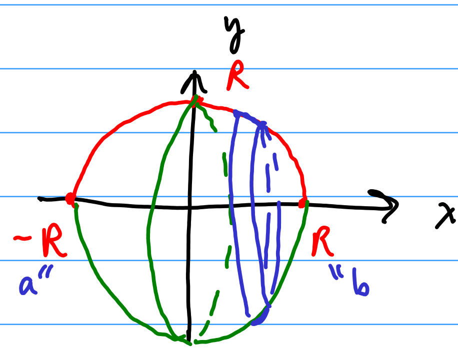
$A = b \cdot h$

$$SA = \lim_{N\to\infty} \sum_{k=1}^{N} 2\pi f(x_k)\cdot ds_k = \int_a^b 2\pi f(x)\, ds = \boxed{2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\, dx}$$

Compute!

revolved around x-axis.

Ex. Sphere w/ radius R. Find the surface area.



$x^2+y^2=R^2$ $\quad y^2 = R^2 - x^2$

$y = \sqrt{R^2-x^2} \leftarrow f(x)$

$$A = \int_{-R}^{R} 2\pi \sqrt{R^2-x^2}\, ds$$

ds: $f'(x) = \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{R^2-x^2}}$, $\quad f'(x)^2 = \dfrac{x^2}{R^2-x^2}$, $\quad 1+f'(x)^2 = \dfrac{R^2-x^2}{R^2-x^2} + \dfrac{x^2}{R^2-x^2}$

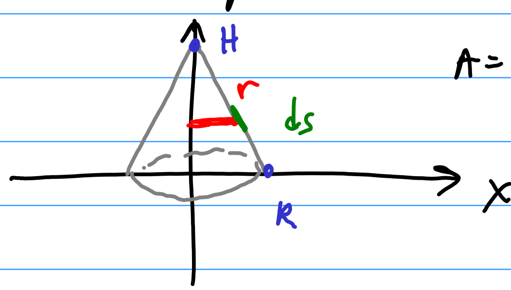
$$1+f'(x)^2 = \frac{R^2}{R^2-x^2} \Rightarrow ds = \sqrt{1+f'(x)^2}\,dx = \frac{R}{\sqrt{R^2-x^2}}\,dx$$

So, $A = 2\pi \int_{-R}^{R} \cancel{\sqrt{R^2-x^2}}\,\frac{R}{\cancel{\sqrt{R^2-x^2}}}\,dx$

$$= 2\pi R \int_{-R}^{R} 1\,dx = 2\pi R\, x\Big|_{-R}^{R} = 2\pi R\left(R-(-R)\right)$$

$$\boxed{A = 4\pi R^2}$$ ✱

Ex. Cone w/ base radius R, and height H.
Right circular



$$A = \int_0^H 2\pi f(y)\,ds(y)$$

$f(y)$: line
$\left.\begin{matrix}(0,H)\\(R,0)\end{matrix}\right\}$

$y = mx+b = -\frac{H}{R}x + H$

$y - H = -\frac{H}{R}x$

$x = \underbrace{-\frac{R}{H}y + R}_{f(y)}$

So $f(y) = \frac{-R}{H}y + R$

$f'(y) = -\frac{R}{H}$ $\qquad f'(y)^2 = \frac{R^2}{H^2}$

$$1+f'(y)^2 = \frac{H^2}{H^2} + \frac{R^2}{H^2} = \frac{H^2+R^2}{H^2}$$

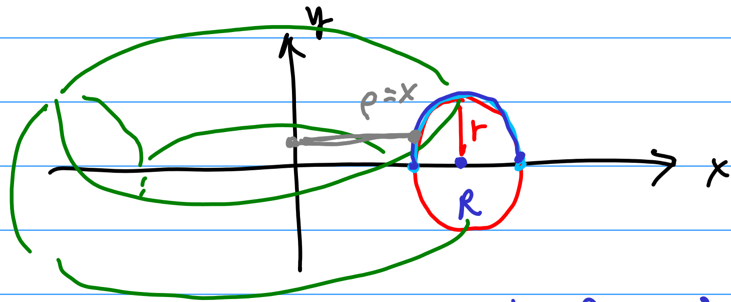
$$ds = \sqrt{1+f'(y)^2}\,dy = \frac{1}{H}\sqrt{H^2+R^2}\,dy$$

$$A = \int_0^H 2\pi\left(-\frac{R}{H}y + R\right)\frac{1}{H}\sqrt{H^2+R^2}\,dy = \frac{2\pi\sqrt{H^2+R^2}}{H}\int_0^H -\frac{R}{H}y + R\,dy$$

$$= \frac{2\pi}{H}\sqrt{H^2+R^2}\left(\left[\frac{-R}{2H}y^2 + Ry\right]_0^H\right) = \frac{2\pi}{H}\sqrt{H^2+R^2}\left(\frac{-R}{2H}H^2 + RH\right)$$

$$\boxed{A = \pi \frac{R}{H}\sqrt{H^2+R^2}}$$

Ex. Torus w/ major radius R and minor radius r.



Find surface area.

$x: R-r \to R+r$

$(x-R)^2 + y^2 = r^2$

$y = \sqrt{r^2 - (x-R)^2}$ ←

$$\frac{1}{2}A = \int_{R-r}^{R+r} 2\pi x \, ds$$

$$\frac{dy}{dx} = \frac{-\cancel{2}(x-R)}{\cancel{2}\sqrt{r^2-(x-R)^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x-R)^2}{r^2-(x-R)^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{r^2 - \cancel{(x-R)^2} + \cancel{(x-R)^2}}{r^2-(x-R)^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \frac{r}{\sqrt{r^2-(x-R)^2}}\, dx$$

$$\frac{1}{2}A = 2\pi r \int_{R-r}^{R+r} \frac{x}{\sqrt{r^2-(x-R)^2}}\, dx = 2\pi r \int_{-r}^{r} \frac{u+R}{\sqrt{r^2-u^2}}\, du$$

$u = x-R$ $\quad$ $x = u+R$ $\quad$ $u(R+r) = r$

$du = dx$ $\quad$ $u(R-r) = -r$

$$\frac{1}{2}A = \underbrace{2\pi r \int_{-r}^{r} \frac{u}{\sqrt{r^2-u^2}}\, du}_{f(-u)=-f(u)\;\; 0} + 2\pi r \int_{-r}^{r} \frac{R}{\sqrt{r^2-u^2}}\, du = 2\pi r R \int_{-r}^{r} \frac{1}{\sqrt{r^2-u^2}}\, du$$

$$= 4\pi r R \int_{0}^{r} \frac{1}{\sqrt{r^2 - u^2}}\, du = 4\pi r R \int_{0}^{\pi/2} \frac{1}{\sqrt{r^2 - r^2\sin^2\theta}}\, r\cos\theta\, d\theta = 4\pi r R \int_{0}^{\pi/2} 1\, d\theta$$

$u = r\sin\theta$ $\qquad$ $\sin\theta = \frac{u}{r}$ $\qquad$ $\theta = \arcsin\left(\frac{u}{r}\right)$

$du = r\cos\theta\, d\theta$ $\qquad$ $\theta(r) = \arcsin\left(\frac{r}{r}\right) = \arcsin(1) = \pi/2$

$\theta(0) = \arcsin\left(\frac{0}{r}\right) = \arcsin(0) = 0$

$$\frac{1}{2} A = 4\pi r R \left(\pi/2\right) = 2\pi^2 r R = \pi r \cdot 2\pi R$$

$$\boxed{A = 2\pi r \cdot 2\pi R}$$