M243 - Calc II

Midterm: 9-10 Oct 2018

Part I: 6-10 Integrals } Ch 6-8

Part II: Applications }

— Review Guide on web page

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§8.5 - Probability

Let f be a function satisfying:

1. f is defined and continuous on $\mathbb{R} = (-\infty, \infty)$
2. $f(x) \geq 0$ for all $-\infty < x < \infty$
3. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Then f is a probability density function (PDF).

$P_f(a \leq X \leq b) = \int_a^b f(x)\,dx$ { probability that X is between a and b.

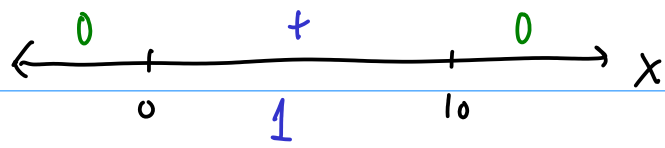
X = continuous random variable

Ex. $f(x) = \begin{cases} 0.006x(10-x) & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$

verify that f is PDF.

1. Cont: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0.006x(10-x) = 0 \cdot 10 = 0$ ✓

$\lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} 0.006x(10-x) = 0.06 \cdot (10-10) = 0$ ✓

2. 

$f(1) = 0.006(1)(10-1) = 0.054 > 0$

3. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{0} 0\,dx} + \int_{0}^{10} 0.006x(10-x)\,dx + \cancel{\int_{10}^{\infty} 0\,dx}$

$$= \int_0^{10} 0.06x - 0.006x^2\,dx = 0.006\int_0^{10} 10x - x^2\,dx$$

$$= 0.006\left(5x^2 - \frac{1}{3}x^3\right)\Big|_0^{10} = 0.006\left(500 - \frac{1000}{3}\right)$$

$$= \frac{500}{3}\left(\frac{6}{1000}\right) = 1$$

Ex. $P(4 \le X \le 8) = ?$

$$\int_4^8 f(x)\,dx = 0.006\left(5x^2 - \tfrac{1}{3}x^3\right)\Big|_4^8$$

$$= \frac{6}{1000}\left(320 - \frac{512}{3} - 80 + \frac{64}{3}\right)$$

$$= \frac{6}{1000}\left(240 - \frac{448}{3}\right)$$

$$= \frac{6}{1000}\left(\frac{720}{3} - \frac{448}{3}\right) = \frac{6}{1000}\left(\frac{272}{3}\right)$$

$$= \cdots = 0.288$$

Ex. $f(t) = \begin{cases} 0 & t < 0 \\ Ae^{-ct} & t \ge 0 \end{cases}$ c fixed constant, $c > 0$

Find A s.t. f is a PDF.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = \overset{0}{\cancel{\int_{-\infty}^{0} 0\,dt}} + \int_{0}^{\infty} Ae^{-ct}\,dt$$

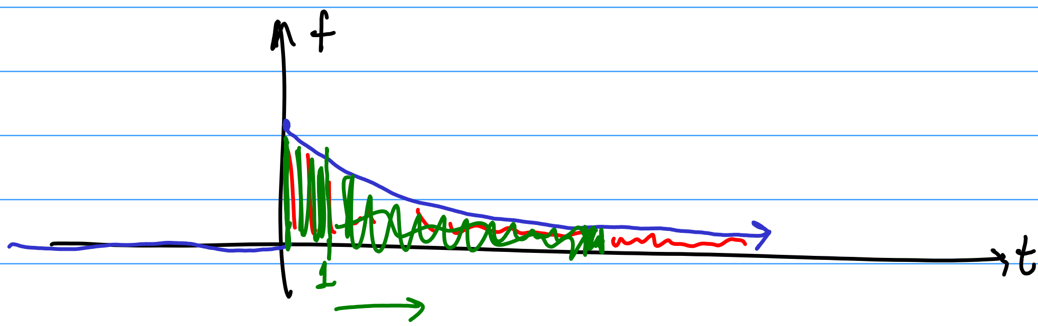
$$= \lim_{u\to\infty} \underbrace{\int_{0}^{u} Ae^{-ct}\,dt} = \lim_{u\to\infty}\left(\frac{A}{-c}e^{-ct}\Big|_{t=0}^{t=u}\right)$$

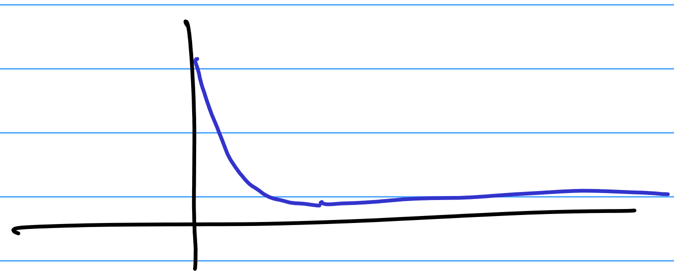
$$1 = \lim_{u\to\infty}\left(\underbrace{\frac{A}{-c}e^{-cu}}_{0} - \frac{A}{-c}e^{0}\right) = 0 + \frac{A}{c}$$

So $\frac{A}{c} = 1$ and $A = c$

We get a PDF of the form:

$$\boxed{f(t) = \begin{cases} 0 & t < 0 \\ ce^{-ct} & t \geq 0 \end{cases} \qquad c > 0}$$





<u>Defn</u>. Let f be a PDF. The <u>mean</u> of the probability distribution is

$$\mu = \bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx$$

The median is the number m such that

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = 1/2$$

Ex. $f(t) = \begin{cases} 0 & t<0 \\ ce^{-ct} & t \geq 0 \end{cases}$ Find μ. (mean)

$$\mu = \int_0^\infty ct\, e^{-ct}\, dt$$

$$= \lim_{u\to\infty} \underbrace{\int_0^u ct\,e^{-ct}\,dt} = \lim_{u\to\infty}\left[-te^{-ct} + \int_0^u \cancel{t}e^{-ct}\,dt\right]$$

$$u = t \qquad dV = ce^{-ct}\,dt$$
$$du = dt \qquad V = -e^{-ct}$$

$$= \lim_{u\to\infty}\left[-te^{-ct} \cancel{+} - \frac{1}{c}e^{-ct}\Big|_{t=0}^{u}\right]$$

$$= \lim_{u\to\infty}\left[\underline{-ue^{-cu}} + 0 \cancel{+} - \frac{1}{c}e^{-cu} + \frac{1}{c}\right]$$

$$= 0 + 0 + 0 + 1/c$$

$$\boxed{\mu = 1/c} \rightarrow c = 1/\mu$$

We can write

$$f(t) = \begin{cases} 0 & t<0 \\ \frac{1}{\mu}e^{-t/\mu} & t \geq 0 \end{cases}$$

Ex. Find m (median) for $f(t) = \begin{cases} 0 & t<0 \\ 0.2e^{-t/5} & t \geq 0 \end{cases}$]

so $\mu = 5$

$$\int_m^{\infty} f(t)\,dt = 1/2 \quad \text{or} \quad \underbrace{\int_{-\infty}^{m} f(t)\,dt = 1/2}$$

$$\int_{-\infty}^{m} f(t)\,dt = \int_0^{m} \frac{1}{5} e^{-t/5}\,dt = 1/2$$

$$\frac{1}{5}\cdot\frac{-5}{1} e^{-t/5} = -e^{-t/5}\Big|_0^m = -e^{-m/5} + e^0$$

$$1 - e^{-m/5} = 1/2$$

so, $e^{-m/5} = 1/2$ $\quad = \ln(2^{-1}) = -\ln(2)$

$$-\frac{m}{5} = \ln(1/2) = -\ln 2$$

$$= \underbrace{\cancel{\ln 1}}_{0} - \ln 2$$

$$+\frac{m}{5} = +\ln 2$$

$$\underbrace{m = 5\ln 2} = \ln(2^5) \approx 3.5$$

$$\boxed{m = \mu \ln 2}$$

(Almost) Probability Distribution $\quad f(x) = e^{-x^2}$

we get a Gaussian Dist., or Normal Distribution