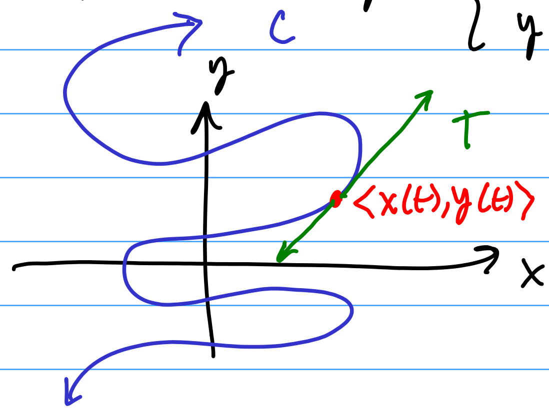
Ch 10 – Parametric Curves and Polar Coordinates

Let C be a curve parametrized by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$



We can try to find the tan. lines.

Need: $dy/dx = \boxed{\frac{dy/dt}{dx/dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

compute

Notation: $\frac{dx}{dt} = \dot{x}$ derivative wrt parameter $\ddot{x} = (\dot{x})^{\cdot}$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ Be careful.

Ex. $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$ Find the tan line at any t.

$\boxed{\vec{r}(t) = \langle t^2, t^3 - 3t \rangle}$ $\dot{\vec{r}}(t) = \langle \overset{\dot{x}}{2t}, \overset{\dot{y}}{3t^2 - 3} \rangle$

$\boxed{\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3t^2 - 3}{2t}}$

At what pts is tan. line horizontal / vertical?

horizontal: $\dot{y} = 0$ vertical: $\dot{x} = 0$

$\dot{y} = 3t^2 - 3 = 3(t^2-1) = 0 \qquad t = \pm 1$

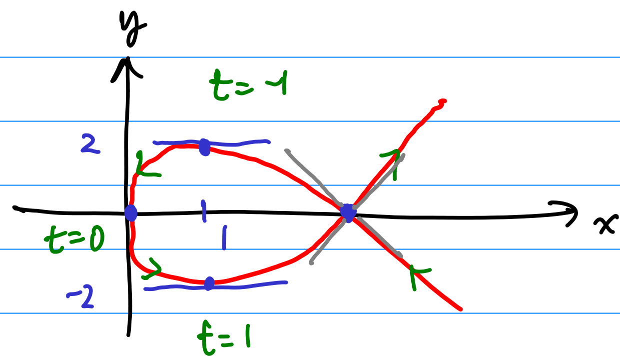
$\dot{x} = 2t = 0 \qquad t = 0$

$\vec{r}(t) = \langle t^2, t^3 - 3t \rangle$

hor. $\vec{r}(1) = \langle 1, -2 \rangle$

$\vec{r}(-1) = \langle 1, 2 \rangle$

vertical: $\vec{r}(0) = \langle 0, 0 \rangle$



Find the intersection pt: $0 = y = t^3 - 3t$

$t(t^2-3) = 0$

$t = 0 \quad t = \pm\sqrt{3}$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} \qquad \left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{3(\sqrt{3})^2-3}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} > 0$$

$$\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \dots = -\sqrt{3} < 0$$

$\frac{dy}{dx}$ represents the velocity of a particle moving along the curve.

$$\frac{d^2y}{dx^2} \neq \frac{\ddot{y}}{\ddot{x}}$$

because $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\left(\dot{y}/\dot{x}\right)^{\cdot}}{\dot{x}}$$

but $\left(\frac{\dot{y}}{\dot{x}}\right)^{\cdot} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)$ requires a quotient rule.

$$= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^2}$$

All together, $$\boxed{\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}} \quad (*)$$

So, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3t^2-3}{2t}\right) = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$\frac{d}{dt}\left(\frac{3}{2}t - \frac{3}{2}\frac{1}{t}\right) = \frac{3}{2} + \frac{3}{2t^2} = \frac{3}{2}\left(1 + 1/t^2\right)$$

$$= \frac{3}{2}\left(\frac{t^2+1}{t^2}\right)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{3}{4}\,\frac{t^2+1}{t^3}}$$

<u>RE</u>. Check that this is the same as using the formula we derived.