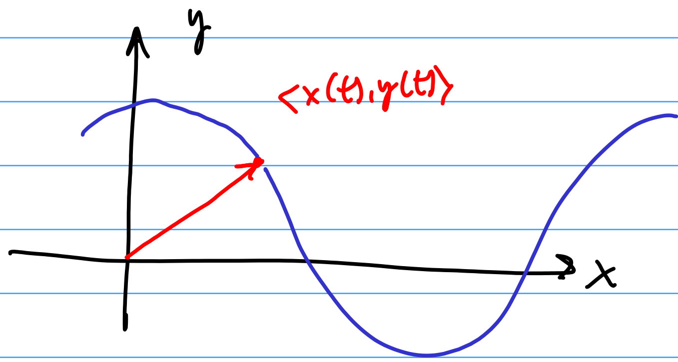
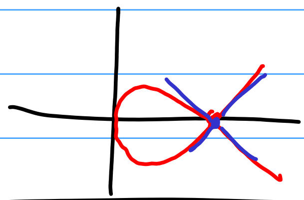
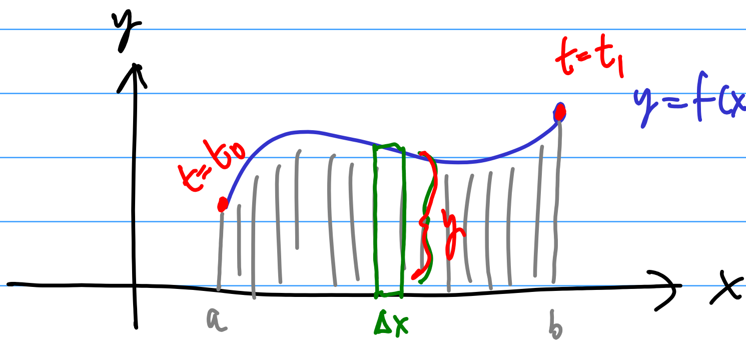
# Ch10 - Parametric Curves




Ex.



## Area under a parametric curve.

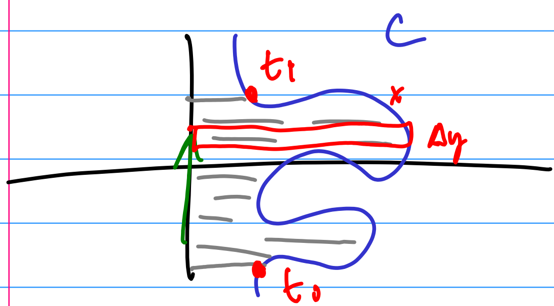



$$y = f(x) = \langle x(t), y(t)\rangle$$

$$A = \int_{a \,(t_0)}^{b \,(t_1)} \underbrace{f(x)}_{y(t)} \underbrace{dx}$$

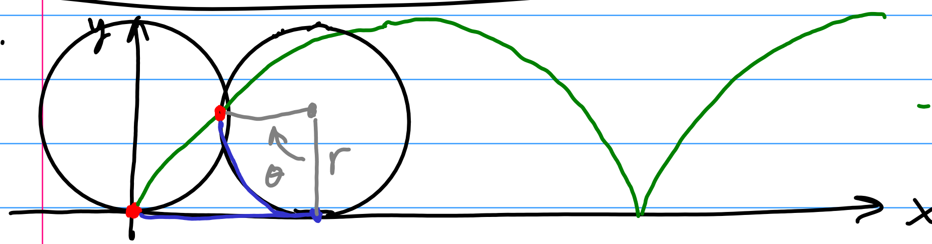
$$\frac{dx}{dt} = \dot{x}(t) \longrightarrow dx = \dot{x}(t)\, dt$$

$$\boxed{A = \int_{t_0}^{t_1} y(t)\, \dot{x}(t)\, dt} \quad (*)$$




$$A = \int_{t_0}^{t_1} x(t)\,\dot{y}(t)\,dt$$

Ex.

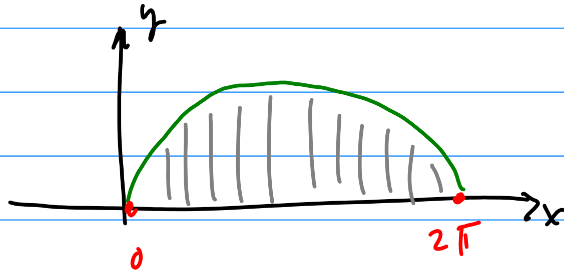



Cycloid

$\theta : 0 \to 2\pi$

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{cases}$$

Find the area under one cycle. w/ $r=1$.




$$A = \int_0^{2\pi} y\,\dot{x}\,d\theta$$

$y(\theta) = (1-\cos\theta)$

$x(\theta) = \theta - \sin\theta$

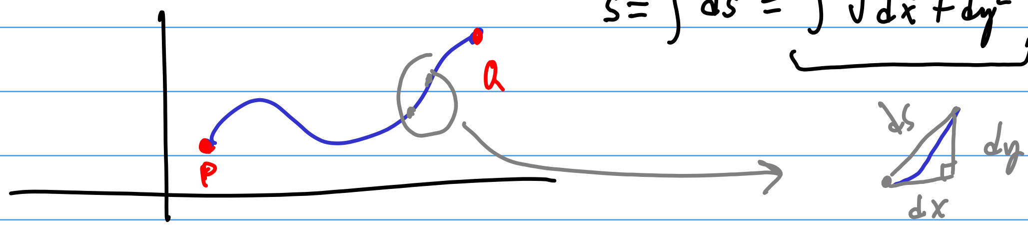
$\frac{dx}{d\theta} = 1 - \cos\theta$

$$A = \int_0^{2\pi} (1-\cos\theta)^2\,d\theta = \int_0^{2\pi} \cos^2\theta - 2\cos\theta + 1 \;d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2\theta) - 2\cos\theta + 1 \;d\theta$$

$$= \frac{3}{2}\theta + \frac{1}{4}\sin(2\theta) - 2\sin\theta \Big|_0^{2\pi} = 3\pi - 0 + 0 - 0 + 0 - 0 = \boxed{3\pi}$$

Arc length?




$$s = \int ds = \underbrace{\int \sqrt{dx^2 + dy^2}}$$

$P(t = t_0)$ $\qquad$ $Q(t = t_1)$

$\langle x(t), y(t) \rangle$

$$\left.\begin{aligned} dx &= \dot{x}\, dt \\ dy &= \dot{y}\, dt \end{aligned}\right\} \quad \begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{(\dot{x}\, dt)^2 + (\dot{y}\, dt)^2} \\ &= \sqrt{((\dot{x})^2 + (\dot{y})^2)\, dt^2} \end{aligned}$$

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$$

$$\boxed{s = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\ dt}$$

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Ex. $\begin{cases} x(\theta) = \theta - \sin\theta \\ y(\theta) = 1 - \cos\theta \end{cases}$ $\qquad 0 \le \theta \le 2\pi$

$\dot{x}(\theta) = 1 - \cos\theta$ $\qquad$ $\dot{y}(\theta) = \sin\theta$

$\dot{x}^2 = (1 - \cos\theta)^2$ $\qquad$ $\dot{y}^2 = \sin^2\theta$

$\quad = \cos^2\theta - 2\cos\theta + 1$

$$ds = \sqrt{\underline{\cos^2\theta} - 2\cos\theta + 1 + \underline{\sin^2\theta}}\ d\theta$$

$$ds = \sqrt{2}\sqrt{1 - \cos\theta}\ d\theta \qquad \text{T.B.I.} \qquad 0 \le \theta \le 2\pi$$

$$s = \int_0^{2\pi} \sqrt{2}\,\sqrt{\underline{1-\cos\theta}}\; d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \leftarrow$

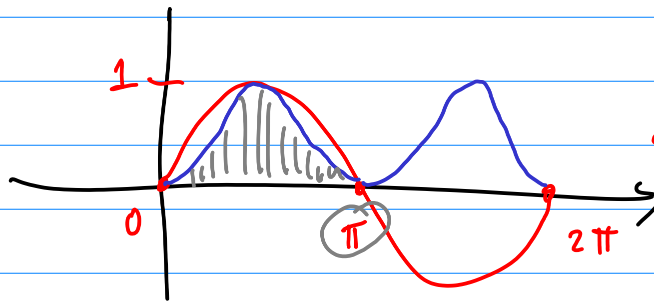
$$= \int_0^{2\pi} \sqrt{2}\,\sqrt{2\sin^2(\tfrac{1}{2}\theta)}\; d\theta$$

$2\sin^2\theta = 1-\cos 2\theta$

$2\sin^2(\tfrac{1}{2}\theta) = 1-\cos\theta$

$$= 2\int_0^{2\pi} \left|\sin(\tfrac{1}{2}\theta)\right| d\theta$$

$$= 2\int_0^{2\pi} \sin(\tfrac{1}{2}\theta)\; d\theta$$




$$= 2\cdot 2\left(-\cos(\tfrac{1}{2}\theta)\right)\Big|_0^{2\pi}$$

$$= 4\left(-(-1)+(1)\right) = 4\cdot 2 = \boxed{8 = s}$$

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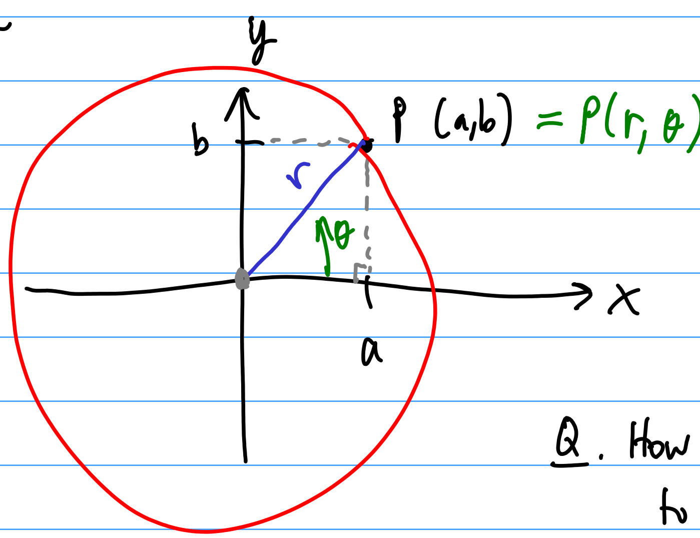
<u>Polar Coordinates</u>




<u>Q</u>. How can $(a,b)$ be related to $(r,\theta)$?