M243 19 Oct 18

Polar curve: $r = r(\theta)$
$$\begin{cases} x = r\cos\theta = r(\theta)\cos\theta \\ y = r\sin\theta = r(\theta)\sin\theta \end{cases}$$

The slope of the tangent line at any pt is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dot{y}}{\dot{x}}$$

$$\dot{y} = \frac{d}{d\theta}\left(r(\theta)\sin\theta\right) = \dot{r}\sin\theta + r\cos\theta$$

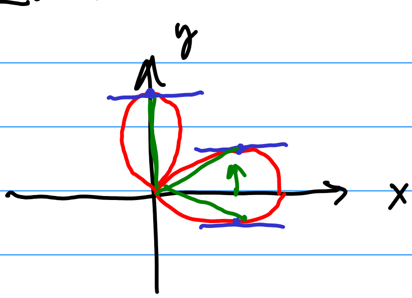
$$\dot{x} = \frac{d}{d\theta}\left(r(\theta)\cos\theta\right) = \dot{r}\cos\theta - r\sin\theta$$

$$\left.\frac{dy}{dx}\right|_\theta = \frac{\dot{r}(\theta)\sin\theta + r(\theta)\cos\theta}{\dot{r}(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{\dot{y}}{\dot{x}}$$

Horizontal: $\dot{y} = 0$ but $\dot{x} \neq 0$

Vertical: $\dot{x} = 0$ but $\dot{y} \neq 0$

Ex. $r = \cos(2\theta)$ Find all pts where tan. line is horizontal.



Parametrization: $\begin{cases} x = \cos(2\theta)\cos\theta \\ y = \cos(2\theta)\sin\theta \end{cases}$

$$\dot{y} = \frac{d}{d\theta}\left(\cos(2\theta)\sin\theta\right)$$
$$= -2\sin(2\theta)\sin\theta + \cos(2\theta)\cos\theta = 0$$

Find θ.

$$\cos(2\theta)\cos\theta = 2\underbrace{\sin(2\theta)}_{2\sin\theta\cos\theta}\sin\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\cos\theta - 2\sin^2\theta\cos\theta = 4\sin^2\theta\cos\theta$$

$$\cos\theta = 6\sin^2\theta\cos\theta$$

$$6\sin^2\theta\cos\theta - \cos\theta = 0$$

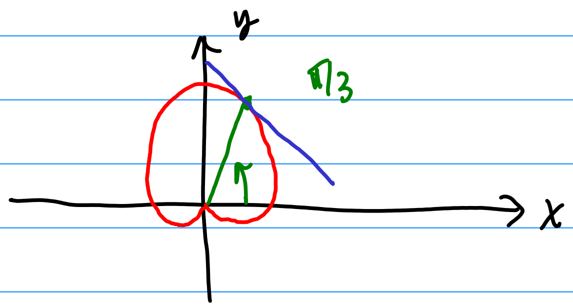
$$\cos\theta\,(6\sin^2\theta - 1) = 0$$

$\cos\theta = 0$, $\theta = \pi/2,\ 3\pi/2$ $\qquad \sin^2\theta = 1/6 \Rightarrow \sin\theta = \pm\sqrt{\frac{1}{6}}$ ← Non-unit-circle :(

use WolframAlpha.
(PTTS)

Ex. $r(\theta) = 1 + \sin\theta$ cardioid

Find dy/dx when $\theta = \pi/3$



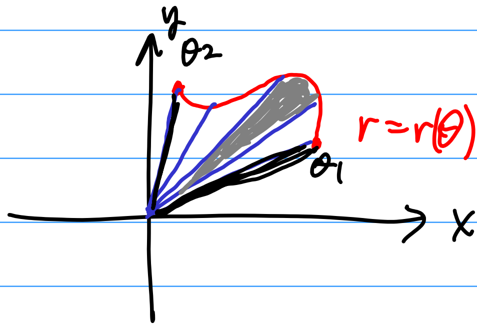
$$\frac{dy}{dx} = \left.\frac{\dot r\sin\theta + r\cos\theta}{\dot r\cos\theta - r\sin\theta}\right|_{\theta=\pi/3} = \frac{\frac{\sqrt3}{\cancel{2}}\dot r(\pi/3) + \frac{1}{\cancel{2}} r(\pi/3)}{\frac{1}{\cancel{2}}\dot r(\pi/3) - \frac{\sqrt3}{\cancel{2}} r(\pi/3)}$$

$$r(\theta) = 1+\sin\theta \qquad \xrightarrow{\theta=\pi/3} \qquad r(\pi/3) = 1 + \sqrt3/2 = \frac{2+\sqrt3}{2}$$
$$\dot r(\theta) = \cos\theta \qquad\qquad\qquad \dot r(\pi/3) = 1/2$$

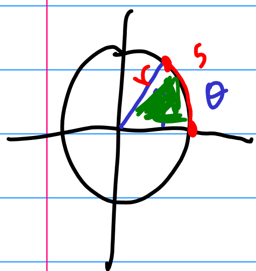
$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\sqrt3(1) + 1(2+\sqrt3)}{1(1) - \sqrt3(2+\sqrt3)} = \frac{2+2\sqrt3}{-(2+2\sqrt3)} = -1$$

§10.4 – Areas and Arc lengths of polar curves

$r = r(\theta)$



$$\boxed{A = \int_{\theta_1}^{\theta_2} \frac{1}{2}(r(\theta))^2 \, d\theta} \quad \circledast$$

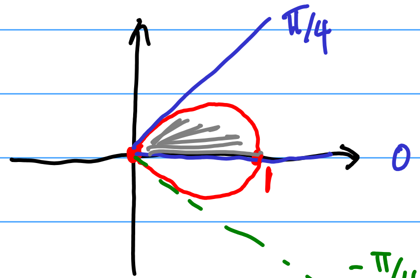


$A = \frac{1}{2} r^2 \theta$ ← in radians.

$\theta = \frac{s}{r} = \frac{4 \cancel{cm}}{5 \cancel{cm}} = \frac{4}{5}$ rad.

Ex. Find the area of 1 leaf of the 4LR.

$r(\theta) = \cos(2\theta)$



$$\cancel{\tfrac{1}{2}} A = \int_0^{\pi/4} \cancel{\tfrac{1}{2}} \cos^2(2\theta)\, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left(1 + \cos(4\theta)\right) d\theta$$

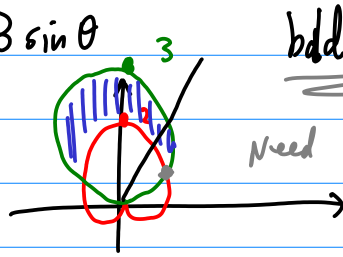
$$= \frac{1}{2} \int_0^{\pi/4} 1 + \cos(4\theta)\, d\theta$$

$$= \frac{1}{2}\left(\pi/4\right) + \frac{1}{8} \sin(4\theta) \Big|_0^{\pi/4}$$

$$= \boxed{\pi/8}$$

Ex. Cardiod: $r = 1 + \sin\theta$ Find the area of the region bdd between these.

circle: $r = 3\sin\theta$

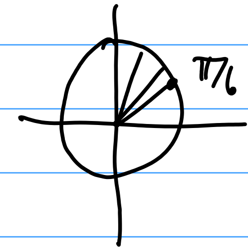


Need intersection θ

set equal, solve for θ.

$$1+\sin\theta = 3\sin\theta \Rightarrow \sin\theta = +1/2$$

$$\theta = \pi/6$$

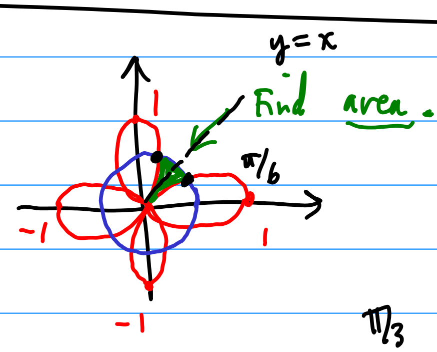


$$\cancel{\tfrac{1}{2}} A = \int_0^{\pi/6} \cancel{\tfrac{1}{2}}\left((3\sin\theta)^2 - (1+\sin\theta)^2 \right) d\theta$$

$$= \int_0^{\pi/6} -1 - 2\sin\theta + 8\sin^2\theta \; d\theta = \dots \text{ FTIS} \dots = \boxed{\pi}$$

Ex. $r = \cos(2\theta)$

$r = 1/2$



$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left(\frac{1}{2}\right)^2 - \cos^2(2\theta) \; d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{1}{4} - \cos^2(2\theta) \, d\theta$$

$$= \frac{3\sqrt{3}-\pi}{48}$$

Find θ_1, θ_2: $\quad \cos(\underbrace{2\theta}) = 1/2$

in QI

$\cos(u) = 1/2 \Rightarrow u = \pi/3$

$2\theta = \pi/3$

$\theta_1 = \pi/6$

$\theta_2 = \pi/2 - \pi/6 = \frac{2\pi}{6} = \pi/3$