Ch 10 Arc length of Polar Curves.

for any parametrized curve, $s = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

$r = r(\theta) \longrightarrow \begin{cases} x = r\cos\theta = x(\theta) = r(\theta)\cos\theta \\ y = r\sin\theta = y(\theta) = r(\theta)\sin\theta \end{cases}$

Compute $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, d\theta$

$$\dot{x} = \frac{dx}{d\theta} = \frac{d}{d\theta}\left(r(\theta)\cos\theta\right)$$

$$= \dot{r}(\theta)\cos\theta - r(\theta)\sin\theta$$

$$(\dot{x})^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+(\dot{y})^2 = \dot{r}^2\sin^2\theta + 2r\dot{r}\cos\theta\sin\theta + r^2\cos^2\theta$$

$$\boxed{\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2}$$

$$\dot{y} = \frac{d}{d\theta}\left(r(\theta)\sin\theta\right) = \dot{r}\sin\theta + r\cos\theta$$

we get:

$$\boxed{s = \int_{\theta_1}^{\theta_2} \sqrt{\dot{r}^2 + r^2}\, d\theta}$$

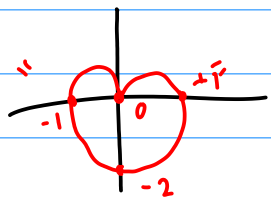
Ex. circumference of circle w/ radius R.

polar function: $r = R > 0 \quad \dot{r} = \frac{d}{d\theta}(R) = 0$

$$ds = \sqrt{r^2 + \dot{r}^2}\, d\theta = \sqrt{R^2 + 0^2}\, d\theta = R\, d\theta$$

$$s = \int_0^{2\pi} R\, d\theta = R \cdot \theta \Big|_{\theta=0}^{2\pi} = 2\pi R$$

Ex. Find the perimeter of the cardioid: $r(\theta) = \sin\theta - 1$



$$\dot{r} = \frac{d}{d\theta}[\sin\theta - 1] = \cos\theta$$

$$r^2 = \sin^2\theta - 2\sin\theta + 1$$
$$+\ \dot{r}^2 = \cos^2\theta$$

$$r^2 + \dot{r}^2 = -2\sin\theta + 2 = 2 - 2\sin\theta$$

$$s = \int_0^{2\pi} \sqrt{2 - 2\sin\theta}\, d\theta = \sqrt{2} \underbrace{\int_0^{2\pi} \sqrt{1 - \sin\theta}\, d\theta}$$ EW.

$$= \sqrt{2} \int_0^{2\pi} \frac{1 - \sin\theta}{\sqrt{1 - \sin\theta}}\, d\theta$$ ~~computer~~

can be done! using trig

$$= \sqrt{2} \cdot 4\sqrt{2} = 4 \cdot 2 = 8$$

Ex. Find arc length of an ellipse w/ "radii" a and b

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$u \qquad v$

$u^2 + v^2 = 1$

$$\begin{cases} u = \cos\theta \\ v = \sin\theta \end{cases} \Rightarrow \begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

Arc length must be done using: $S = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\, d\theta$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta$$

$$= \underline{\text{FTIS}}.$$

Tomorrow – Conic Sections!