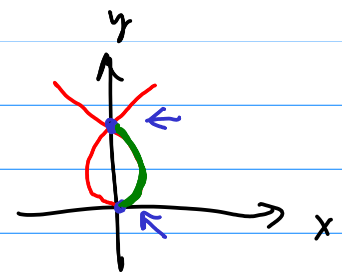
14. Question Details

Find the length of the loop of the given curve.

$x = 9t - 3t^3, \quad y = 9t^2$

$\vec{r}(t) = \langle 9t - 3t^3, 9t^2 \rangle$



$x = 0: \quad 9t - 3t^3 = 0$

$-3t(t^2 - 3) = 0$

$t = 0 \qquad t = \pm\sqrt{3}$

$$\frac{1}{2} s = \int_0^{\sqrt{3}} ds = \int_0^{\sqrt{3}} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

$$\dot{x} = \frac{d}{dt}(9t - 3t^3) = 9 - 9t^2 = 9(1 - t^2)$$

$$\dot{x}^2 = 81(1-t^2)^2 = 81(1 - 2t^2 + t^4)$$

$$\dot{y} = \frac{d}{dt}(9t^2) = 2 \cdot 9t$$

$$\dot{y}^2 = 4 \cdot 81 t^2$$

$$\dot{x}^2 + \dot{y}^2 = 81(1 - 2t^2 + t^4) + 81 \cdot 4t^2 = 81(1 + 2t^2 + t^4)$$

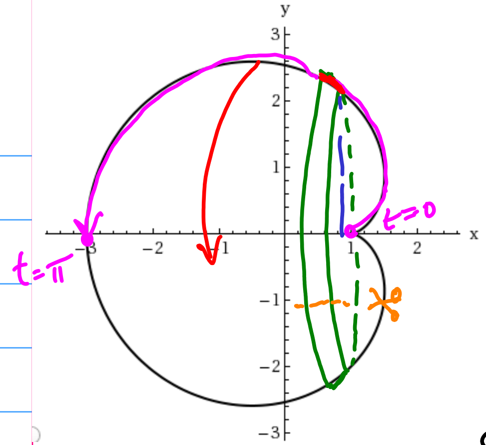
$$= 9^2(1+t^2)^2 = (9(1+t^2))^2$$

So $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \sqrt{(9(1+t^2))^2}\, dt = 9(1+t^2)\, dt$

and,

$$s = 2 \cdot 9 \int_0^{\sqrt{3}} 1 + t^2 \, dt = 18\left(t + \frac{1}{3}t^3\right)\Big|_{t=0}^{\sqrt{3}}$$

$$= 18\left(\sqrt{3} + \frac{3\sqrt{3}}{3}\right) = 36\sqrt{3}$$



Find SA of rotated solid.

$2\pi y$ ds

$$SA = 2\pi \int_0^{\pi} y\, ds = 2\pi \int_0^{\pi} y \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

$x = 2\cos t - \cos(2t)$ $\qquad \dot{x} = -2\sin t + 2\sin(2t)$

$y = 2\sin t - \sin(2t)$ $\qquad \dot{y} = 2\cos t - 2\cos(2t)$

$$(\dot{x})^2 = 4\sin^2 t - 8\sin t \sin(2t) + 4\sin^2(2t)$$

$$+\ (\dot{y})^2 = 4\cos^2 t - 8\cos t\cos(2t) + 4\cos^2(2t)$$

$$4 - 8(\sin t\sin(2t) + \cos t\cos(2t)) + 4$$

$$= 8(1 - (\sin t\sin(2t) + \cos t\cos(2t)))$$

$$8(1 - \underbrace{(2\sin^2 t\cos t + \cos t(1 - 2\sin^2 t))}_{\cos t})$$

$$ds = \sqrt{8(1-\cos t)}\, dt$$

so $$SA = 2\pi \int_0^{\pi} (2\sin t - \underbrace{\sin(2t)}_{2\sin t\cos t}) \sqrt{8(1-\cos t)}\, dt$$

$$= 4\sqrt{8}\,\pi \int_0^{\pi} \sin t\,(1-\cos t)\sqrt{(1-\cos t)}\, dt$$

$$= 4\sqrt{8}\,\pi \int_0^{\pi} (\underbrace{1-\cos t}_{u})^{3/2} \underbrace{\sin t\, dt}_{du}$$

$u = 1 - \cos t$

$du = \sin t\, dt$

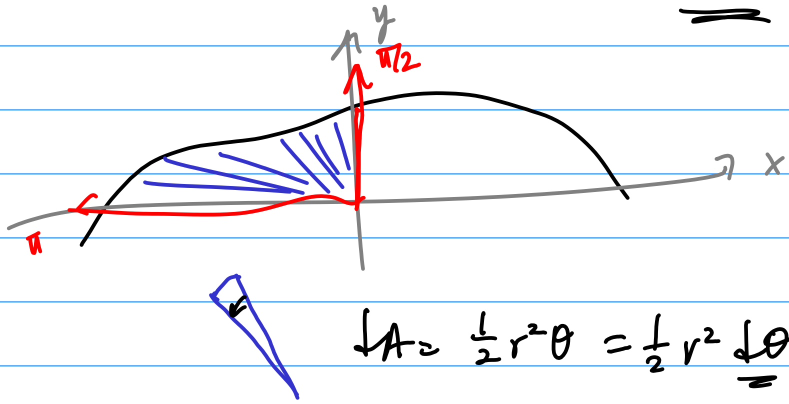
$u(\pi) = 1 - (-1) = 2$

$u(0) = 1 - 1 = 0$

$$= 8\sqrt{2}\,\pi \int_0^2 u^{3/2}\, du = 8\sqrt{2}\,\pi \left(\tfrac{2}{5} u^{5/2} \Big|_0^2\right)$$

$$8\sqrt{2}\pi \cdot \frac{2}{5} \cdot 4\sqrt{2} = \boxed{\frac{128}{5}\pi}$$

10.4.#) Find the area bdd by $r(\theta) = e^{-\theta/14}$, $\frac{\pi}{2} \le \theta \le \pi$



$$dA = \frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \, d\theta$$

$$A = \int_{\pi/2}^{\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{\pi/2}^{\pi} \left(e^{-\theta/14}\right)^2 d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} e^{-\theta/7} \, d\theta$$

$$= \frac{1}{2} \left(-7 e^{-\theta/7} \Big|_{\pi/2}^{\pi} \right)$$

$$= \frac{7}{2} \left(e^{-\pi/14} - e^{-\pi/7} \right)$$