M243 25 Oct 18

Ch 9. Differential Equations

Let $y = y(x)$ is an unknown function of x.

A differential equation for y is an equation relating y and (some of) its derivatives.

Ex. $dx\,\frac{dy}{dx} = 2y\,dx$ TBD: solve for y.

$$\frac{1}{y}\,dy = \frac{2y\,dx}{y} \rightarrow \int \frac{1}{y}\,dy = \int 2\,dx$$

$$e^{\ln|y|} = e^{2x + C}$$

$$y(x) = e^{2x+C} = e^{C}e^{2x} = Ce^{2x}$$

$$\boxed{y(x) = Ce^{2x}}$$ solution

$$\frac{dy}{dx} = \frac{d}{dx}\left[Ce^{2x}\right] = 2\cdot\underbrace{Ce^{2x}} = 2y$$

Ex. $\frac{dy}{dx} = x^2 \rightarrow \int dy = \int x^2\,dx$

$$\boxed{y = \frac{1}{3}x^3 + C}$$

Ex. $\frac{dy}{dx} = 5y + \frac{3}{2}$

$$dy = 5\left(y + \frac{3}{10}\right) dx \rightarrow \int \frac{1}{y+3/10}\, dy = \int 5\, dx$$

$$\ln\left(y + \frac{3}{10}\right) = 5x + C$$

$$e \qquad e$$

$$y + \frac{3}{10} = Ce^{5x}$$

$$\boxed{y(x) = Ce^{5x} - \frac{3}{10}}$$

Linear Differential Equations

let $p(x)$ and $g(x)$ be continuous functions

A linear DE is of the form

$$\frac{dy}{dx} + p(x)\, y = g(x)$$

Ex. $xy' + y = 2x \rightarrow y' + \frac{1}{x} y = 2$ $\qquad p(x) = 1/x \leftarrow x \neq 0$, $g(x) = 2$

$$\frac{dy}{dx} + \frac{1}{x} y = 2$$

$$\frac{dy}{dx} = 2 - \frac{1}{x} y = y\left(\frac{2}{y} - \frac{1}{x}\right) = \frac{1}{x}(2x - y)$$

Instead, suppose $\underline{\mu(x) > 0}$. Then

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right]$$ has the same solution as our original DE.

$$\boxed{\mu(x)\frac{dy}{dx} + \mu(x)p(x)\,y} = \mu(x)\,g(x)$$

Want: $\dfrac{du}{dx} = \dfrac{d}{dx}\left[\mu(x)\,y(x)\right] = \boxed{\mu(x)\dfrac{dy}{dx} + \dfrac{d\mu}{dx}\,y(x)}$

Now compare

$$\mu(x)\frac{dy}{dx} + \boxed{\mu(x)p(x)}\,y = \mu(x)\,g(x)$$

and

$$\mu(x)\frac{dy}{dx} + \boxed{\frac{d\mu}{dx}}\,y = \mu(x)\,g(x)$$

For this to work, then μ must satisfy,

$$\frac{d\mu}{dx} = \mu p$$

We can solve it!

$$\int \frac{1}{\mu}\,d\mu = \int p(x)\,dx$$

$$\ln \mu = \int p(x)\,dx$$

so $\boxed{\mu(x) = e^{\int p(x)\,dx}}$ Integrating Factor

So we obtain $\dfrac{dy}{dx} + p(x)\,y = g(x)$ is equivalent to

$$\frac{d}{dx}\left[\mu(x)\,y(x)\right] = \mu(x)\,g(x)$$

$$\int d\left[\mu(x)\,y(x)\right] = \int \mu(x)g(x)\,dx$$

$$\mu(x)\,y(x) = \int \mu(x)g(x)\,dx + C$$

$\circledast$ $\boxed{y = \dfrac{1}{\mu}\displaystyle\int \mu g\,dx + \dfrac{C}{\mu}}$ w/ $\boxed{\mu = e^{\int p\,dx}}$

Ex. $\frac{dy}{dx} + \frac{1}{x} y = 2$ $\quad p(x) = 1/x \leftarrow$ find μ

$g(x) = 2$

$$\mu = e^{\int p\,dx} = e^{\int 1/x\,dx} = e^{\ln x} = x$$

$$y = \frac{1}{\mu}\int \mu g\,dx + \frac{c}{\mu} = \frac{1}{x}\underbrace{\int x \cdot 2\,dx} + \frac{c}{x}$$

$p(x) = 1/x,\ x \neq 0$

$$= \frac{1}{x}\left(x^2\right) + \frac{c}{x}$$

$$\boxed{y = x + \frac{c}{x}}$$

Ex. $\frac{dy}{dx} + 3x^2 y = 6x^2$ $\quad p(x) = 3x^2 \leftarrow$ $\quad \mu = e^{\int p\,dx}$

$g(x) = 6x^2$

$$\mu = e^{\int 3x^2 dx} = \boxed{e^{x^3}}$$

$$y = \frac{1}{\mu}\int \mu g\,dx + \frac{C}{\mu} = \frac{1}{e^{x^3}}\underbrace{\int \overset{2\cdot 3}{6} x^2 e^{x^3}\,dx} + \frac{C}{e^{x^3}}$$

$\cancel{u \quad \frac{dv}{e^{x^3}}}$

$u = x^3$

$du = 3x^2\,dx$

$$= e^{-x^3}\int 2 \cdot e^u\,du + Ce^{-x^3}$$

$$= e^{-x^3} \cdot 2 \cdot e^{x^3} + Ce^{-x^3}$$

$$\boxed{y = 2 + Ce^{-x^3}}$$