$y = y(x)$ unknown (function) of x.

$$\frac{dy}{dx} = f(x,y)$$

$$\frac{dy}{dx} = -p(x)\,y + g(x) \qquad \text{Linear}$$

$$\frac{dy}{dx} = M(x)\cdot N(y) \qquad \text{separable}$$

$$\int \frac{1}{N(y)}\,dy = \int M(x)\,dx$$

... then, if possible, solve for y as a function of x.

Ex. $\dfrac{dy}{dx} = \dfrac{x^2}{(1-y^2)}$

$$\int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

$$\boxed{-\tfrac{1}{3}y^3 + y - \tfrac{1}{3}x^3 = C}$$

for every choice of C we get a different solution curve.

Ex. $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$ $\qquad y(0) = -1 \leftarrow$ Initial value

$$(2y-2)\,dy = (3x^2+4x+2)\,dx$$

$$\int_{-1}^{y} 2t-2\,dt = \int_0^x 3u^2+4u+2\,du$$

$$t^2-2t\Big|_{-1}^{y} = u^3+2u^2+2u\Big|_0^x$$

$$(y^2-2y)-((-1)^2-2(-1)) = x^3+2x^2+2x-0$$

$$y^2-2y-3 = x^3+2x^2+2x$$ Solve for y.

$$y^2-2y+1 = \underbrace{x^3+2x^2+2x+3}_{A}+1$$

$$(y-1)^2 = A+1$$

$$y = 1\pm\sqrt{A+1} = 1\pm\sqrt{x^3+2x^2+2x+4}$$

$$y(0) = 1\pm\sqrt{0^3+2\cdot0^2+2\cdot0+4} = 1\pm\sqrt{4} = 1\pm2 = -1$$

keep − sign

$$y = 1-\sqrt{x^3+2x^2+2x+4}$$

Ex. $x\,dx + ye^{-x}\,dy = 0$

$$x\,dx = -ye^{-x}\,dy$$

$$\int xe^x\,dx = -\int y\,dy$$

$$xe^x-e^x+C = -\tfrac{1}{2}y^2$$

$$y^2 = -2xe^x+2e^x+C$$

$$\boxed{y = \pm\sqrt{-2xe^x+2e^x+C}}$$

if $y(0)=2$, then $y=?$

$$2 = y(0) = \pm\sqrt{\underbrace{-\cancel{2\cdot 0 e^0} + 2\cdot e^0 + C}_{=4}}$$

$2+C=4$

$C=2$

$$\boxed{y = \sqrt{-2xe^x + 2e^x + 2}}$$

Ex. Homogeneous Equation

$$\frac{dy}{dx} = \frac{y-4x}{x-y} \qquad \text{Not separable.}$$

Make a change of variable: $\boxed{v(x) = y/x}$ $\Rightarrow$ $y = xv(x)$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{y/x - 4}{1 - y/x} = \frac{v-4}{1-v}$$

$$-v + v + x\frac{dv}{dx} = \frac{v-4}{1-v} \quad \frac{-v(1-v)}{1-v}$$

SEPARABLE! $\left[\; x\frac{dv}{dx} = \frac{v^2-4}{1-v}\right.$

$$\int \frac{1-v}{v^2-4}\,dv = \int \frac{1}{x}\,dx$$

$$-\tfrac{1}{2}\ln|v^2-4| + \tfrac{1}{2}\arcsin\left(\tfrac{v}{2}\right) = \ln|x| + C$$

Aside:

$u = v^2-4$

$du = 2v\,dv$

$$\frac{1}{2}\int \frac{-v\cdot 2}{v^2-4}\,dv + \int \frac{1}{v^2-4}\,dv$$

$$-\tfrac{1}{2}\ln|v^2-4| \quad \cancel{+\tfrac{1}{2}\arcsin\left(\tfrac{v}{2}\right)} \quad -\tfrac{1}{2}\operatorname{arctanh}\left(\tfrac{v}{2}\right)$$

SHENAN'S

$$\boxed{-\frac{1}{2}\ln\left(\frac{y^2}{x^2}-4\right) + \cancel{\frac{1}{2}\arcsin\left(\frac{y}{2x}\right)} = \ln(x) + C} \quad (*)$$

$-\frac{1}{2}\operatorname{arctanh}\left(\frac{y}{2x}\right)$