M243 — 30 Oct 2018

§9.3–9.5 – Separable and Linear DE

1. Find the general solutions of the DE:

a.) $dy/dx = 3x^2y^2,\ y \neq 0$

b) $y' - y = 3e^x$

c.) $\dfrac{du}{dt} = \dfrac{3+t^4}{t^2u + t^2u^4}$

d.) $dy + 3e^{x+y}\,dx = 0$

e.) $t\ln t\,\dfrac{dr}{dt} + r = 3te^t$

f.) $xy' + y = 13\sqrt{x}$

2. Solve the initial value problems (IVP):

a.) $\begin{cases} \dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u} \\ u(0) = -4 \end{cases}$

b.) $\begin{cases} x\ln x = y\left(1+\sqrt{8+y^2}\right)\dfrac{dy}{dx} \\ y(1) = 1 \end{cases}$

c.) $\begin{cases} t^3\dfrac{dy}{dt} + 3t^2y = 7\cos t \\ y(\pi) = 0 \end{cases}$

d.) $\begin{cases} t\dfrac{du}{dt} = t^2 + 3u,\ t>0 \\ u(5) = 100 \end{cases}$

e.) $\begin{cases} y' = \dfrac{2\sin x}{\sin y} \\ y(1) = \pi/2 \end{cases}$

Brief Solutions

1. a.) $dy/dx = 3x^2y^2,\ y \neq 0$

Separable!

$$\int \frac{1}{y^2}\,dy = \int 3x^2\,dx$$

$$-1/y = x^3 + C$$

$$\boxed{y = \frac{-1}{x^3+C}}$$

b) $y' - y = 3e^x$

Linear! $p(x) = -1$, $\mu(x) = e^{-\int 1\,dx} = e^{-x}$

$$y = 1/\mu \int \mu g\,dx + C/\mu$$

$$y = e^x \int 3\cancel{e^x}\,\cancel{e^{-x}}\,dx + Ce^x$$

$$\boxed{y = 3xe^x + Ce^x}$$

c.) $\dfrac{du}{dt} = \dfrac{3+t^4}{t^2u + t^2u^4}$ Separable!

$$\frac{du}{dt} = \frac{3+t^4}{t^2(u+u^4)} \to \int (u+u^4)\,du = \int \left(\frac{3}{t^2} + t^2\right)dt$$

$$\frac{1}{2}u^2 + \frac{1}{5}u^5 = \frac{-3}{t} + \frac{1}{3}t^3 + C$$

$$\boxed{15u^2 + 6u^5 + 90\tfrac{1}{t} - 10t^3 = C}$$

d.) $dy + 3e^{x+y}\,dx = 0$

Separable!

$$-dy = 3e^xe^y\,dy$$

$$\int -e^{-y}\,dy = \int 3e^x\,dx$$

$$e^{-y} = 3e^x + C$$

$$\boxed{y = -\ln|3e^x + C|}$$

e.) $t\ln t\,\dfrac{dr}{dt} + r = 3te^t$ Linear!

$$dr/dt + \frac{1}{t\ln t}r = \frac{3e^t}{\ln t}$$

$p = \dfrac{1}{t\ln t}$, $\int p\,dt = \int \dfrac{1}{t\ln t}\,dt = \ln(\ln t)$

So, $\mu = e^{\ln(\ln t)} = \ln t$, and

$$y = \frac{1}{\ln t}\int \frac{3e^t}{\cancel{\ln t}}\cancel{\ln t}\,dt + \frac{C}{\ln t}$$

$$\boxed{y = \frac{3e^t}{\ln t} + \frac{C}{\ln t}}$$

f.) $xy' + y = 13\sqrt{x}$ Linear!

$$y' + 1/x\,y = \frac{13}{\sqrt{x}} \qquad p = 1/x,\ \mu = e^{\int 1/x\,dx} = e^{\ln x} = x$$

$$y = 1/x \int 13\sqrt{x}\,dx + C/x$$

$$y = 1/x \cdot 13 \cdot \frac{2}{3}x^{3/2} + C/x$$

$$\boxed{y = \frac{26}{3}\sqrt{x} + C/x}$$

2. a) $\begin{cases} \frac{du}{dt} = \frac{2t + \sec^2 t}{2u} \\ u(0) = -4 \end{cases}$

Separable!

$2u\,du = (2t + \sec^2 t)\,dt$

$\int_{-4}^{u} 2x\,dx = \int_0^t 2y + \sec^2 y\,dy$

$x^2\Big|_{-4}^{u} = \left[y^2 + \tan y\right]_0^t$

$u^2 - 16 = t^2 + \tan t$

$u^2 = t^2 + \tan t + 16$

$\boxed{u = -\sqrt{t^2 + \tan t + 16}}$

b) $\begin{cases} x \ln x = y\left(1 + \sqrt{8 + y^2}\right)\frac{dy}{dx} \\ y(1) = 1 \end{cases}$

Separable!

$\int x \ln x\,dx = \int y + y\sqrt{8+y^2}\,dy$

$u = \ln x$ $\quad dv = x\,dx$ $\quad$ | $\quad u = 8 + y^2$

$du = \frac{1}{x}dx$ $\quad v = \frac{1}{2}x^2$ $\quad$ | $\quad du = 2y\,dy$

$\frac{1}{2}x^2 \ln x - \frac{1}{6}x^2 + C = \frac{1}{2}y^2 + \frac{1}{3}\sqrt{8+y^2}^{\,3}$

$\frac{1}{2}\cdot 0 - \frac{1}{6} + C = \frac{1}{2} + 9 \rightarrow C = \frac{29}{3}$

$\boxed{\frac{1}{2}y^2 + \frac{1}{3}\sqrt{8+y^2}^{\,3} + \frac{1}{6}x^2 - \frac{1}{2}x^2 \ln x = \frac{-29}{3}}$

c) $\begin{cases} t^3 \frac{dy}{dt} + 3t^2 y = 7\cos t \\ y(\pi) = 0 \end{cases}$

Linear!

$\frac{dy}{dt} + \frac{3}{t}y = \frac{7}{t^3}\cos t, \quad p = \frac{3}{t}, \mu = e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$

$y = t^{-3}\int \cancel{t^3} \cdot \frac{7}{\cancel{t^3}}\cos t\,dt + Ct^{-3}$

$y = t^{-3}\sin t + Ct^{-3}$

$y(\pi) = 0 + C\pi^{-3} = 0 \Rightarrow C = 0$

So, $\boxed{y = \frac{\sin t}{t^3}}$

d.) $\begin{cases} t\frac{du}{dt} = t^2 + 3u, \; t > 0 \\ u(5) = 100 \end{cases}$

Linear!

$\frac{du}{dt} - \frac{3}{t}u = t, \quad p = \frac{-3}{t}, \mu = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}$

$u = t^3 \int t^{-3} \cdot t\,dt + Ct^3$

$u = t^3 \cdot \left(-\frac{1}{t}\right) + Ct^3$

$u = -t^2 + Ct^3$

$u(5) = -25 + 125C = 100 \Rightarrow C = 1$

so $\boxed{u(t) = t^3 - t^2}$

e.) $\begin{cases} y' = \frac{2\sin x}{\sin y} \\ y(0) = \pi/2 \end{cases}$

Separable!

$\sin y\,dy = 2\sin x\,dx$

$\int_{\pi/2}^{y} \sin u\,du = \int_0^x 2\sin t\,dt$

$-\cos u\Big|_{\pi/2}^{y} = -2\cos t\Big|_0^x$

$+\cos y + (0) = +2\cos x - 2(1)$

$\cos y = 2\cos x - 2$

$\boxed{y = \arccos(2\cos x - 2)}$