M243 7 Nov'18

§11.2 – Series

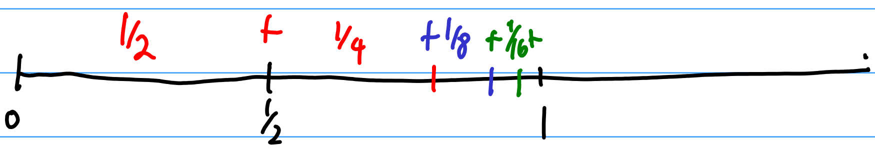
Let $\{a_n\}$ be a sequence $\{a_n\} = \{a_0, a_1, a_2, \dots\}$.
The <u>series</u> corresponding to $\{a_n\}$ is

$$S = a_0 + a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{i=0}^{\infty} a_i$$

<u>Q</u>. Does this sum make sense? Can we find the sum S?

<u>Ex</u>. $1+2+3+4+5+\dots+n+\dots = +\infty$ <u>Diverges</u>

<u>Ex</u>. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots + \frac{1}{2^n} + \dots = 1$
There's a chance!



$$\sum_{i=1}^{\infty} \frac{1}{2^n} = 1 \quad \text{\underline{convergent}}.$$

<u>Ex</u>. $\displaystyle\sum_{i=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \sum_{i=1}^{\infty} \frac{1}{2^n} = 1 + 1 = 2.$

How to add these in general?

$$S = \sum_{i=1}^{\infty} a_i$$

<u>Partial Sums</u>

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + \dots + a_k$

The partial sums form a sequence $\{S_k\} = \{S_1, S_2, S_3, \dots, S_k, \dots\}$.
The <u>sum</u> of the series, if it exists, is defined to be

$$\sum_{i=1}^{\infty} a_i = S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{i=1}^{k} a_i$$

<u>Ex.</u> <u>Geometric Series</u>

$$S = \sum_{i=1}^{\infty} a r^{i-1}$$

$a =$ constant
$r =$ constant

$$S = a + ar + ar^2 + ar^3 + \dots + ar^k + \dots = \sum_{i=0}^{\infty} ar^i$$

<u>Partial Sum</u>:

$$S_k = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^k}$$
$$- \; rS_k = \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^k} + ar^{k+1}$$

$$S_k - rS_k = a - ar^{k+1}$$
$$(1-r) S_k = a(1-r^{k+1})$$

$$\boxed{S_k = \frac{a(1-r^{k+1})}{1-r}}$$

$$S = \lim_{k\to\infty} \frac{a(1-r^{k+1})}{1-r} = \frac{a\left(1 - \lim_{k\to\infty} r^{k+1}\right)}{1-r}$$

$$\lim_{k\to\infty} r^{k+1} = \begin{cases} 0 & |r| < 1 \\ 1 & r = 1 \\ \infty & |r| > 1 \end{cases}$$

<u>Thm</u>.

$$\boxed{\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}} \quad \text{if} \quad |r| < 1$$

and $\sum_{i=0}^{\infty} ar^i$ diverges if $|r| \geq 1$.

Proof. $\sum_{i=0}^{\infty} a1^i = \sum_{i=0}^{\infty} a = a+a+a+a+\cdots+a+\cdots = +\infty$

Ex. $\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4^n \frac{3}{3^n} = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n = \sum_{n=1}^{\infty} \cancel{3}\cdot\frac{4}{\cancel{3}}\left(\frac{4}{3}\right)^{n-1} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$

Geometric! $a=4$ $r=\frac{4}{3}>1$

Diverges!

Ex. $f(x)=\sum_{n=0}^{\infty} x^n$

$f(x) = 1+x+x^2+x^3+\cdots+x^n+\cdots$ "Infinite Polynomial"

looks geometric w/ $a=1$, $r=x$

domain = $|x|<1 \Rightarrow -1<x<1$

$S=\frac{a}{1-r}$

$$\boxed{f(x)=\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad -1<x<1}$$

!