M243 8 Nov'18

Ex. $f(x)=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+x^4+\cdots = \frac{1}{1-x}$, $\underline{|x|<1}$

$= \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ $a=1$, $r=x$, $|r|<1$

Ex. $2.3\overline{17} = 2.317171717\ldots = \frac{m}{n}$ Find the fractional representation.

$2.3\overline{17} = 2.31717171717\ldots$

$= 2.3 + 0.017 + 0.00017 + 0.0000017 + 0.000000017 + \cdots$

$= 2.3 + 17(10^{-3}) + 17(10^{-5}) + 17(10^{-7}) + \cdots$

$= 2.3 + \frac{17}{10^3}\cdot 1 + \frac{17}{10^3}\left(\frac{1}{10^2}\right) + \frac{17}{10^3}\left(\frac{1}{10^2}\right)^2 + \frac{17}{10^3}\left(\frac{1}{10^2}\right)^3 + \cdots$

$\uparrow$

$= 2.3 + \sum_{n=0}^{\infty} \overbrace{\left(\frac{17}{10^3}\right)}^{a} \overbrace{\left(\frac{1}{10^2}\right)}^{r}{}^{n}$

$= 2.3 + \frac{a}{1-r} = 2.3 + \frac{17}{1000}\cdot\frac{1}{1-1/100}\cdot\frac{100}{100}$

$= 2.3 + \frac{17\cdot \cancel{100}}{10\cancel{00}\cdot 99} = \frac{23\cdot 99}{10\cdot 99} + \frac{17}{990} = \frac{2277+17}{990} = \frac{2294}{990}$

$\boxed{2.3\overline{17} = \frac{1147}{495}}$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

$S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{k(k+1)}$

Yuck!

Instead, try PFD, $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$1 = A(n+1) + Bn$

$$n=0: \quad 1 = 1A \quad \Rightarrow \quad A=1$$

$$n=-1: \quad 1 = -B \quad \Rightarrow \quad B=-1$$

Let's add 'em:

$$S_k = \sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{k}}-\frac{1}{k+1}\right)$$

"Telescoping"

$$S_k = 1-\frac{1}{k+1} \leftarrow$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right) = 1-0 = 1$$

so $\boxed{\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1}$

Ex. Harmonic Series: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots = +\infty$ Divergent

Reason: $\underbrace{S_{2^k} = \displaystyle\sum_{n=1}^{2^k}\frac{1}{n} > 1+\frac{k}{2}}$ Claim.

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} S_{2^k} > \lim_{k\to\infty}\left(1+\frac{k}{2}\right) = +\infty$$

Thm. If $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

$P\to Q$ means $\neg Q\to\neg P$

Proof. Since $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent, then $\displaystyle\sum_{n=0}^{\infty} a_n = S = \lim_{k\to\infty} S_k$.

$$S_k = a_0+a_1+a_2+\cdots+a_k$$

$$S_{k+1} = a_0+a_1+a_2+\cdots+a_k+a_{k+1}$$

$$a_{k+1} = S_{k+1}-S_k$$

$$\lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty}\left(S_{k+1}-S_k\right) = \lim_{k\to\infty} S_{k+1} - \lim_{k\to\infty} S_k = S-S = 0. \quad \blacksquare$$

The contrapositive of this theorem is the <u>Test for Divergence</u> (TD):

<u>Thm</u>. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

<u>Ex</u>. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ Convergent or Divergent?

<u>TD</u>: $$\lim_{n\to\infty} \frac{n^2 \cdot \frac{1}{n^2}}{(5n^2+4)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{5+4/n^2}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 5 + \lim_{n\to\infty} 4/n^2}$$

$$= \frac{1}{5+0} = 1/5 \neq 0$$

by TD, $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is <u>divergent</u>.

<u>Thm</u>. $\sum a_n, \sum b_n$ <u>convergent</u> implies:

a) $\sum k a_n = k \sum a_n$ for constant k

b) $\sum (a_n \pm b_n) = (\sum a_n) \pm (\sum b_n)$

<u>Ex</u>. $$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 3 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{=1} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{=1}$$

$$= 3 \cdot 1 + 1$$

$$= 4$$