M243 9 Nov '18

§11.3 – Integral Test

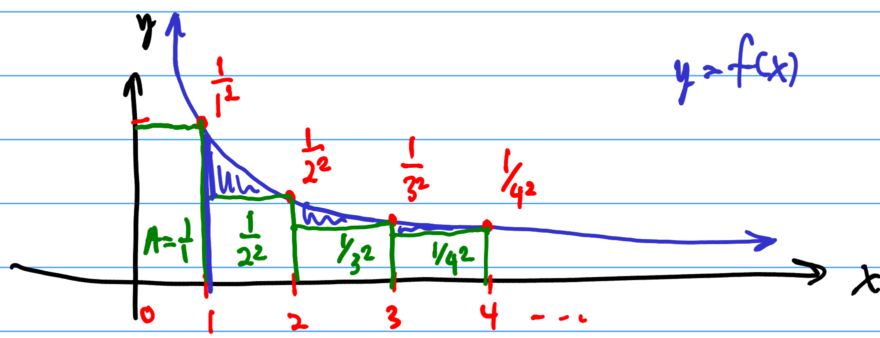
Series : $\sum_{n=1}^{\infty} a_n$

Q. Is the series convergent, or divergent?

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

TD: $\lim_{n\to\infty} \frac{1}{n^2} = 0$ inconclusive.

$f(x) = \frac{1}{x^2}$



$\sum_{n=1}^{\infty} \frac{1}{n^2}$ = Sum of areas of boxes

Area under the curve $f(x) = 1/x^2$ is

$$A = \int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{-1}{x} \Big|_1^t \right)$$

$$= \lim_{t\to\infty} \left(1 - \frac{1}{t} \right) = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2$$

So, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must be convergent, and

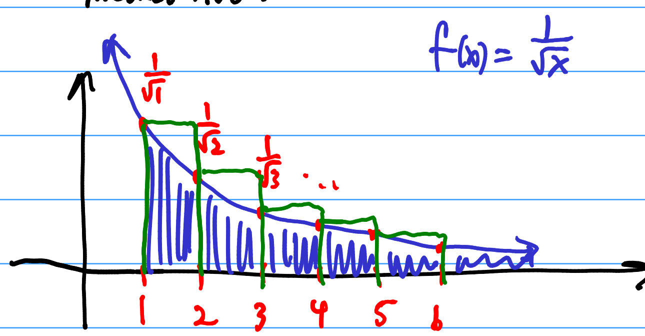
$$1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2.$$

The exact answer is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

<u>Ex.</u> $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ C or D?

<u>TD</u>: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ inconclusive.

$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$



$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$$

$$\int_1^{\infty} x^{-1/2}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/2}\,dx = \lim_{t\to\infty} 2x^{1/2}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(2\sqrt{t} - 2\right) = +\infty \quad \text{Diverges!}$$

So, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent since $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ is divergent.

Thm. <u>Integral Test</u> (IT)
Let $\sum_{n=1}^{\infty} a_n$. Let f be a positive, decreasing function on $[1,\infty)$ such that $f(n) = a_n$ for all $n \geq N$. Then $\sum_{n=1}^{\infty} a_n$ is convergent <u>if and only if</u> $\int_1^{\infty} f(x)\,dx$ is convergent.

$$\begin{cases} 1.) \text{ if } \int_1^\infty f(x)\,dx \text{ is } \underline{\text{convergent}}, \text{ then } \sum_{n=1}^\infty a_n \text{ is also.} \\ 2.) \text{ if } \int_1^\infty f(x)\,dx \text{ is divergent, the so is } \sum_{n=1}^\infty a_n. \end{cases}$$

<u>Ex.</u> $\sum_{n=1}^\infty \frac{1}{n}$ Harmonic Series.

$$\sum_{n=1}^\infty \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{C or D?}$$

IT: $\int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \lim_{x\to\infty} \ln x - \cancel{\ln 1} = +\infty$

Therefore, $\sum_{n=1}^\infty \frac{1}{n}$ diverges!

<u>Corollary</u>. (p-Test, pT) $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent iff $p > 1$.

<u>Proof</u>. $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if and only if $p > 1$. ■

Ex. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ C or D?

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \left\{ \begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array} \right\} = \int_0^{\infty} u\,du = \frac{1}{2}(\infty)^2 - 0 = +\infty$$

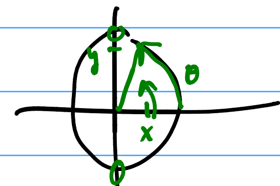
$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent.

Ex. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ C or D?

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx \overset{u}{=} \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \text{Diverges!}$$

so $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$



$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_1^{\infty} = \lim_{x\to\infty} \arctan(x) - \arctan(1)$$

$$= \pi/2 - \pi/4 = \pi/4 \quad \text{converges!}$$

so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent!

$$\frac{1}{n^2+1} \leq \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

convergent by pT

So we guess $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is also. Next week!