M243 12 Nov '18

§11.3 ish. $\sum_{n=1}^{\infty} a_n$ Converge or Diverge?

(IT) $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is convergent, where $f(n)=a_n$, $f>0$, $f'(x)<0$.

(CT) Comparison Test. $0 \le a_n \le b_n$ for all $n \ge N$, and $\sum_{n=1}^{\infty} b_n$ is convergent. Then $\sum_{n=1}^{\infty} a_n$ is also convergent.

OR

$0 \le b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ is divergent. Then $\sum_{n=1}^{\infty} a_n$ is also.

Ex. C or D: $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ (IT): $\int_1^{\infty} \frac{5}{2x^2+4x+3}\,dx$ PFD... "/

CT: Idea: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by pT.

We want: $\frac{5}{2n^2+4n+3} \le b_n$

$$\frac{5}{2n^2+4n+3} = \frac{5}{2}\,\frac{1}{n^2+2n\cancel{+3/2}} \le \frac{5}{2}\,\frac{1}{n^2} \quad \text{for } n \ge 1$$

Then $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \le \sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is convergent.

Hence $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ is convergent by CT.

Ex. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ C or D?

IT: $\int_1^{\infty} \frac{\ln x}{x}\,dx$ $u = \ln x$ $du = \frac{1}{x}dx$ $u(\infty)=\infty$ $u(1)=0$

$= \int_0^{\infty} u\,du$ diverges by pT.

To use CT, we need a b_n s.t. $b_n \le \frac{\ln n}{n}$ for $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ must diverge.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} = \underbrace{\frac{\boxed{\overset{=0}{\ln 1}}}{1}}_{n=1} + \underbrace{\frac{\boxed{\overset{<1}{\ln 2}}}{2}}_{n=2} + \underbrace{\frac{\boxed{\overset{>1}{\ln 3}}}{3} + \frac{\boxed{\overset{>1}{\ln 4}}}{4} + \cdots}_{\boxed{n=3 \to \infty}}$$

We notice that $\frac{1}{n} > \frac{\ln n}{n}$ for $n \ge 3$.

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by pT.

So CT says that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent.

Ex. $\sum_{n=1}^{\infty} \frac{5}{2n^2-4n-3}$ Converge or Diverge?

$$\frac{5}{2n^3} \underset{?}{>} \frac{5}{2n^2-4n-3} > \underbrace{\frac{5}{2n^2}}_{\text{converges by pT}}$$

Thm. Limit Comparison Test

Suppose $a_n, b_n \geq 0$ for $n \geq N$, and consider the series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n.$$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ $(c \neq 0, c \neq \infty)$,

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same convergence.

Back to Ex. $\sum_{n=1}^{\infty} \frac{5}{2n^2-4n-3}$ LCT: $\underbrace{\sum_{n=1}^{\infty} \frac{5}{2n^2}}$

$$\lim_{n\to\infty} \frac{\cancel{5}}{\cancel{2}n^2 - \cancel{4}^{2}n - 3/_{2}} \cdot \frac{\cancel{2}n^2}{\cancel{5}} = \lim_{n\to\infty} \frac{1 \, n^2}{1 \, n^2 - 2n - 3/2} = 1 \neq 0$$ ☺

Since the limit is not 0, then $\sum_{n=1}^{\infty} \frac{5}{2n^2-4n-3}$ is convergent

Since we know that $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is convergent (pT).

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent otherwise } (p\leq 1) \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^p} dx$$

$\underline{Ex}$. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ C or D?

$\underline{TD}$: $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$. DMW.

$\underline{IT}$? $\int_1^{\infty} f(x)\,dx$ $f(n) = \sin\left(\frac{1}{n}\right)$, $f(x) = \sin\left(\frac{1}{x}\right)$

$\left.\begin{matrix} f > 0 \text{ eventually.} \\ f'(x) < 0 \text{ eventually} \end{matrix}\right\} \checkmark$

$\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx = ?$ Too hard.

CT or LCT: what to compare to?

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Try #1. $\sin(n)$, $\sum_{n=1}^{\infty} \sin(n)$ = Diverge.

Compare: $\sin\left(\frac{1}{n}\right) \geq \sin(n)$? CT = no,

LCT: $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\sin(n)} = \frac{0}{\sim\sim} = 0$. Inconclusive.

$\underline{Try\ \#2}$. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ = $\underline{\text{Divergent}}$.

CT: $\sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$ happens to be true, but how do we show it?

OR.

LCT: $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n} = \frac{0}{0}$

$x = \frac{1}{n}$

$$= \lim_{x \to 0^+} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0} \overset{L'H}{=}$$

$$= \lim_{x \to 0^+} \frac{\cos x}{1} = \cos 0^+ = 1 > 0$$

So $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ have the same behavior.

hence,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \text{divergent.}$$