M243 — 13 NOV 18

§11.4 Alternating Series Test (AST)

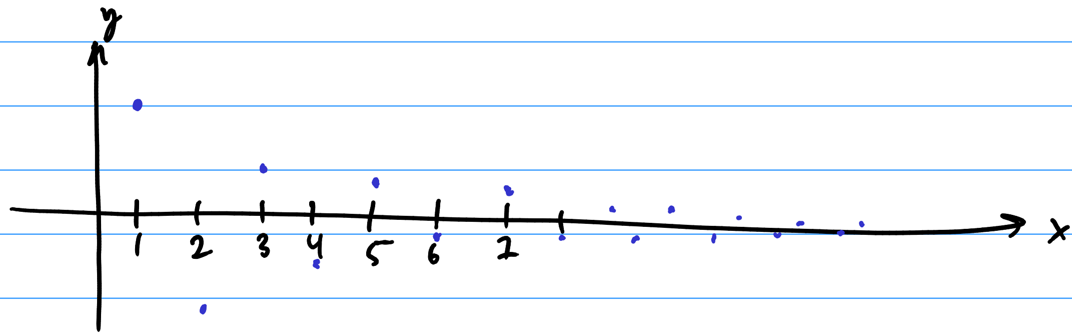
Thm. Let $\sum a_n = \sum (-1)^n b_n = b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + - \cdots$

s.t. 1.) $0 < b_{n+1} \le b_n$ "decreasing"

and 2. $\lim_{n\to\infty} b_n = 0$

Then $\sum a_n = \sum (-1)^n b_n$ is convergent.

Sketch.



Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \cdots$

C or D?

AST:
1.) alternating ✓
2.) decreasing: $\frac{1}{n+1} < \frac{1}{n}$ ✓
3.) limit: $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

Convergent!

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ looks alternating. Now check.

$a_n = (-1)^n b_n$ $\qquad b_n = \frac{3n}{4n-1} > 0$ alternating ✓

Decreasing? $b_{n+1} < b_n$? $\qquad \frac{3(n+1)}{4(n+1)-1} \; \square \; \frac{3n}{4n-1}$?

$$b_n = \frac{3n}{4n-1} \longleftrightarrow f(x) = \frac{3x}{4x-1}$$

$$f'(x) = \frac{(4x-1)3 - 3x(4)}{(4x-1)^2} = \frac{-3}{(4x-1)^2} < 0$$

so f is decreasing, hence b_n are also decreasing. ✓

<u>Limit</u>: $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$

so $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is divergent by <u>TFD.</u>

<u>Ex.</u> $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

<u>Limit</u>: $\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{0n^3 + n^2}{1n^3+1} = \frac{0}{1} = 0$ ✓

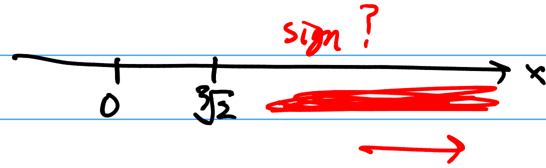
<u>Alternating</u>: $b_n = \frac{n^2}{n^3+1} > 0$ ✓

<u>Decreasing</u>: $b_{n+1} \le b_n$? $\quad b_{n+1} = \frac{(n+1)^2}{(n+1)^3+1} \square \frac{n^2}{n^3+1} = b_n$

$$b_n \longleftrightarrow f(x) = \frac{x^2}{x^3+1}$$

$$f'(x) = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{-x^4 + 2x}{(x^3+1)^2}$$

$$-x^4+2x=0$$
$$-x(x^3-2)=0$$
$$x=0 \qquad x=\sqrt[3]{2}$$



if $\underline{x\geq 2}$ $\quad f'(x)=\dfrac{-x^4+2x}{(x^3+1)^2}<0$

So b_n is <u>eventually</u> decreasing. ✓

Therefore $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^3+1}$ is convergent by AST.

<u>Thm</u>. Alternating Sum Estimate.

Let $\sum a_n$ be an alternating series that converges by the AST.

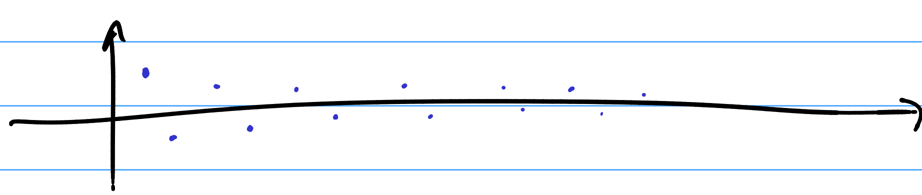
$$\begin{cases} a_n=(-1)^n b_n \\ 0<b_{n+1}\leq b_n \\ \lim\limits_{n\to\infty} b_n=0 \end{cases}$$

Then the remainder $|R_k|=|S-S_k|\leq b_{k+1}$.

$$S=\underbrace{S_k}_{\text{partial sum}}+\underbrace{R_k}_{\text{remainder}}$$

partial sum & remainder

<u>Proof</u>. The sum $S=\sum a_n$ lies between any two adjacent partial sums



So, $|S - S_k| \leq |S_{k+1} - S_k| = b_{k+1}$. ■

Ex. Alt. Harm. Series: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Add this up correct to 3 decimal places. Don't actually do it.
How large should n be to guarantee 3 correct decimal places?

$$|R_n| \leq \frac{1}{1000}$$

$$|R|_n \leq b_{n+1} = \boxed{\frac{1}{n+1} \leq \frac{1}{1000}}$$

$$n+1 \geq 1000$$

choose, $n \geq 1001$

Ex. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - + \cdots$

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$
$$12! = 12(11)(10)(9) 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

Q. How large should we choose n to ensure $|R_n| \leq \frac{1}{1000}$?

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1)!} \leq \frac{1}{1000}$$

$$(n+1)! \geq 1000$$

n	$(n+1)!$
1	$2! = 2$
2	$3! = 6$
3	$4! = 24$
4	$5! = 120$
5	$6! = 720$
6	$7! = 5040 > 1000$

$\boxed{n = 6}$:

$$S_6 = \sum_{n=1}^{6} (-1)^{n+1} \frac{1}{n!}$$

$$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} =$$

$$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} \approx \boxed{0.368}$$

Def'n. A series is said to be <u>absolutely convergent</u> if

$\sum |a_n|$ is convergent.

i.e., $\sum a_n = a_1 + a_2 + \cdots$
$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$

By the Squeeze Theorem, if $\sum a_n$ is <u>absolutely convergent</u>, then it is also <u>convergent</u>.

$$-\sum |a_n| \leq \sum a_n \leq \sum |a_n|,$$

and $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0,$

<u>Ex.</u> $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ C or D?

C	D
2 + 1/2	1/2

$$a_n = \frac{\cos(n)}{n^2}$$

$$|a_n| = \left|\frac{\cos(n)}{n^2}\right| = \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$$

<u>CT:</u> $0 < \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{C by pT.}}$

So $\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$ is convergent by CT/pT.

Therefore $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is <u>absolutely convergent</u>.