M243 14 Nov '18

§11.5 Ratio and Root Tests

Thm Ratio Test (RaT)

Let $\sum a_n$ be any series.

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.

2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or $=\infty$), then $\sum a_n$ diverges.

3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive.

Ex. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

$$|a_n| = \frac{n^3}{3^n}, \qquad |a_{n+1}| = \frac{(n+1)^3}{3^{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{\cancel{3^n}}{n^3} = \frac{1}{3}\,\frac{(n+1)^3}{n^3} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$3^{n+1} = \cancel{3^n}\,3$$

then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^3$

$$= \frac{1}{3}\cdot 1^3 = \frac{1}{3} < 1$$

So, by RaT, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent. ✓

Ex. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\frac{5^5 = 5\cdot5\cdot5\cdot5\cdot5}{5! = 5\cdot4\cdot3\cdot2\cdot1}$$

Guess: Divergent.

RaT: $a_n = \frac{n^n}{n!}$ $\quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty}$$

Let $\ln\left(L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$

$$\ln L = \lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right)$$

$$= \lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n}\right)$$

$$= \infty \cdot \ln 1 = \infty \cdot 0$$

L'H

$$= \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+1/n}\cdot\cancel{(1/n)'}}{\cancel{(1/n)'}} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1$$

So $\ln L = 1$

$$e^{L} = e^{\lim_{n\to\infty}}\left(1+\frac{1}{n}\right)^n = e > 1$$

So, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$ So $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ diverges! by RaT.

Thm. Root Test (RoT)

$\sum a_n$

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent.

2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum a_n$ diverges.
   $(= \infty)$

3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, RoT is inconclusive.

Ex. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ C or D?

$$|a_n| = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \left(\left(\frac{2n+3}{3n+2}\right)^n\right)^{1/n} = \frac{2n+3}{3n+2} \xrightarrow{n\to\infty} \frac{2}{3} < 1$$

So $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent by RoT.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ For what values of p is this series convergent?

Abs. Conv. : $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^p}\right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ abs. conv. for $p > 1$.

But conditional convergence, $p \leq 1$.

$$\sum a_n = \sum (-1)^{n-1} b_n \rightarrow b_n = \frac{1}{n^p}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$$

AST: 1. $0 \le b_{n+1} \le b_n$

$$f(x) = \frac{1}{x^p} = x^{-p}$$

$$f'(x) = -px^{-p-1} = \frac{-p}{x^{p+1}} \qquad x \ge 1$$

For $f(x)$ to be decreasing on $[1, \infty)$

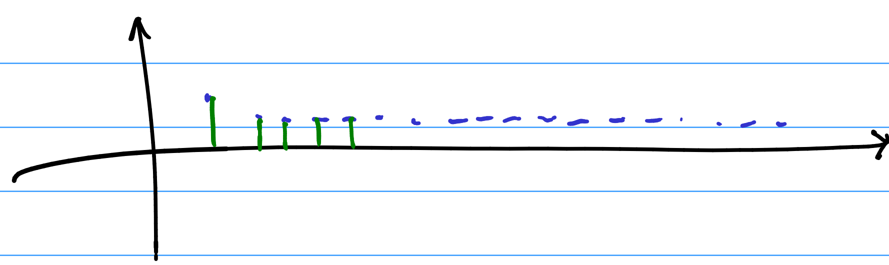
$$-p < 0 \Rightarrow \boxed{p > 0}$$

2. $\lim b_n = 0$?

$$\lim \frac{1}{n^p} = \frac{1}{\infty^p} = \frac{1}{\infty} = 0 \text{ for } p > 0.$$

So, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is conditionally convergent for $0 < p \le 1$

and divergent for $p \le 0$.



Ex. Srinivasa Ramanujan (~1910)

$$\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + 26390n)}{(n!)^4 \; 396^{4n}} = \frac{1}{\pi}$$

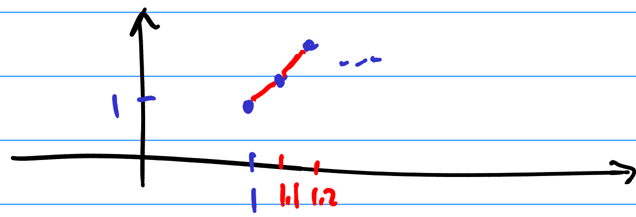
~1985: William Gosper used this series to find the first 17 million digits of π. ☺

Apply RaT (FTIS).

9.2 #8. $y' = 3y + 2xy = f(x,y)$ $\quad y(1.5) = ?$

$y(1) = 1$

$h = 0.1$



$x_0 = 1$

$h = 0.1$

y_k = known

$$y_{k+1} = y_k + f(x_0 + h\cdot k, y_k)\cdot h$$

k	x_k	y_k
0	1	1
1	1.1	$1 + (3\cdot 1 + 2\cdot 1\cdot 1)\cdot 0.1 = 1 + 0.5 = 1.5$
2	1.2	$1.5 + (3(1.5) + 2(1.1)(1.5))\cdot 0.1 =$