M243 20 Nov 18

§11.8 Power Series

Defn. A power series is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $\{a_n\} = \{a_0, a_1, a_2, \dots\}$ are the coefficients and x_0 is fixed called the center of the series.

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

Ex. $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$

$$f(0) = 1 + 0 + 0^2 + 0^3 + \cdots = 1 + 0 = 1$$

The domain of f is the set of all x for which the series $f(x)$ converges.

$$f(1) = \sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + 1 + \cdots = +\infty \quad \text{so } 1 \text{ is not in dom.}$$

$$f(\tfrac{1}{2}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

$f(x) = \sum_{n=0}^{\infty} x^n$ is geometric w/ $a = 1$, $r = x$

So f is convergent for $|x| < 1$

and the domain of f is $-1 < x < 1$.

Def'n. $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

Interval of Convergence (IC) : domain of f

check endpoints.

$|x-x_0| < R$

Radius of Convergence : R

Ex. $f(x) = \sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + \cdots$

$f(0) = 1 + 0 \cdots = 1.$

RaT: $\left| \frac{(n+1)!\ x^{n+1}}{n!\ \ x^n} \right| = \left| \frac{x^{\cancel{n+1}}}{\cancel{x^n}} \right| \cdot \left| \frac{(n+1)\cdot \cancel{n!}}{\cancel{n!}} \right| = |x|\,(n+1)$

$\lim_{n\to\infty} |x|\,(n+1) = |x| \lim_{n\to\infty} (n+1) = \infty > 1$ for all x, $x \neq 0$.

So $\sum_{n=0}^{\infty} n!\, x^n$ diverges for all $x \neq 0$.

So, $R = 0$. domain is just $x = 0$.

Ex. $f(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ find IC and RC.

RaT: $\left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| = |x-3| \left(\frac{n}{n+1} \right) \xrightarrow{n\to\infty} \boxed{|x-3| < 1}$ Convergence

R

by RaT, the domain is

Include the endpoints?

$-1 < x-3 < 1$

$+3 \quad +3 \quad +3$

$2 < x < 4$

$x=4$: $\quad f(4) = \sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by pT

So 4 is <u>not</u> included.

$x=2$: $\quad f(2) = \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - + \cdots$

$b_n = \frac{1}{n} \xrightarrow{n\to\infty} 0 \checkmark$

$b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n} \checkmark$

So convergent by AST, and

$x=2$ <u>is</u> in domain.

$R = 1$, and IC: $2 \le x < 4$

<u>Ex.</u> $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{2^{2n}(n!)^2}}_{a_n} (x^2)^n$

RaT: $\left| \frac{a_{n+1} \, x^{2(n+1)}}{a_n \, x^{2n}} \right| = |x|^2 \underbrace{\left| \frac{a_{n+1}}{a_n} \right|}_{\lim_{n\to\infty}} = |x|^2 \cdot L < 1$

$|x|^2 < \underbrace{\frac{1}{L}}_{R^2}$

$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cancel{(-1)^{n+1}}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{\cancel{(-1)^n}} \right| = \frac{\cancel{2^{2n}}\,\cancel{(n!)^2}}{\cancel{2^{2n}}\, 2^2 (n+1)^2 \cancel{(n!)^2}}$

$= \frac{1}{4(n+1)^2} \longrightarrow 0 \quad < 1$

So $R = \infty$, and $IC = \mathbb{R}$.