M243 28 Nov '18

Power Series:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots + \cdots = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

$$\int \sum_{n=0}^{\infty} a_n(x-x_0)^n \, dx =$$

$$= \int a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots + a_n(x-x_0)^n + \cdots \, dx$$

$$= C + a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \cdots + \frac{a_n(x-x_0)^{n+1}}{n+1} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{a_n(x-x_0)^{n+1}}{n+1}$$

Then,
$$\int \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n \right) dx = \sum_{n=0}^{\infty} \left(\int a_n(x-x_0)^n \, dx \right) + C$$

Ex. $\frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n(x-x_0)^n \right]$

$$= \frac{d}{dx}\left[a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots \right]$$

$$= a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots + \underline{na_n(x-x_0)^{n-1}} + \cdots$$

$$= \underbrace{\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}} = \sum_{n=1}^{\infty} \frac{d}{dx}\left[a_n(x-x_0)^n \right]$$

let $m = n-1 \rightarrow m+1 = n$ then $m(1) = 0$

and

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n (x-x_0)^n\right] = \sum_{m=0}^{\infty} (m+1)a_{m+1}(x-x_0)^m \qquad m=n$$

$$= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x-x_0)^n$$

Ex. Bessel Function : $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$\boxed{0! = 1}$

Find $J_0'(x)$

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{64\cdot 36} + - \cdots$$

$$J_0'(x) = \underbrace{0 \quad - \frac{2x}{4}}_{} + \frac{4\cancel{x}^3}{\cancel{6}4} - \frac{6x^5}{64\cdot 36} + - \cdots + \frac{(-1)^n}{2^{2n}(n!)^2}\cdot 2n \cdot x^{2n-1} + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n n x^{2n-1}}{2^{2n-1}(n!)^2}}$$

Ex. $\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n = f(x)\ e^x}$

Compute $f'(x)$.

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n!} n \cdot x^{n-1} = \sum_{n=1}^{\infty} \frac{\cancel{n}\,1}{\cancel{n}\cdot(n-1)!} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1}$$

Compare :

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

So $f'(x) = f(x)$

So, $\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$!

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$$

and $$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots \leftarrow$$

or $$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$:)

Ex. Find the IC (domain) of $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Use RaT: $$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \cdot \frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}} = \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0$$

$0 < 1$ for all x, hence the interval of convergence is $\mathbb{R} = (-\infty, \infty)$

Ex. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ Compute $f''(x)$ and compare w/ $f(x)$.

$$f(x) = \boxed{1} \boxed{- \frac{1}{2!}x^2} \boxed{+ \frac{1}{4!}x^4} - \frac{1}{6!}x^6 + \cdots + \boxed{\frac{(-1)^n x^{2n}}{(2n)!}} + \cdots$$

$$f'(x) = 0 - \frac{1}{2!}2x + \frac{1}{4!}4x^3 - \frac{1}{6!}6x^5 + \cdots \frac{(-1)^n 2n\, x^{2n-1}}{(2n)!} + \cdots$$

$$f''(x) = -\frac{1}{2!}\cdot 2\cdot 1 + \frac{1}{4!}\cdot 4\cdot 3x^2 - \frac{1}{6!}6\cdot 5x^4 + \cdots + \frac{(-1)^n (2n)(2n-1)\, x^{2n-2}}{(2n)!}$$

$$= \boxed{-1} \boxed{+ \frac{1}{2!}x^2} \boxed{- \frac{1}{4!}x^4} + \cdots + \frac{(-1)^n x^{2n-2}}{(2n-2)!} + \boxed{\frac{(-1)^{n+1} x^{2n}}{(2n)!}} + \cdots$$

So, $f''(x) = -f(x)$

it turns out that this is $\cos(x)$.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

FTS: $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$(\sin(x))' = \cos(x) \qquad R = \infty$$

Ex. $f(x) = \frac{1}{(x+1)^2}$ Find a P.S. representation.

Consider $\underbrace{g(x) = \frac{1}{1+x}}$ $\qquad g'(x) = \frac{-1}{(1+x)^2} = -f(x)$

Start w/ $g(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} 1\cdot(-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

IC: $(-1, 1)$

$$g'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right] = \sum_{n=1}^{\infty} (-1)^n n\, x^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)\, x^n \qquad \text{IC: } (-1, 1)$$

So finally $f(x) = \frac{1}{(x+1)^2} = -1 \cdot \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n$

$$\boxed{\frac{1}{(x+1)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n}$$

$(-1,1)$