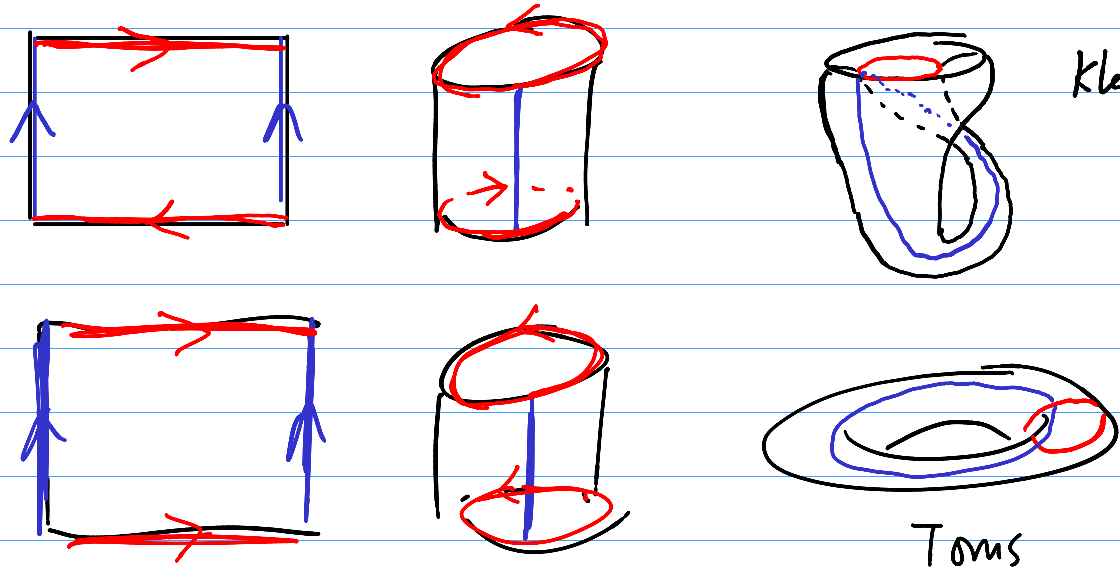


Let f be a function w/ infinitely many derivatives.
Let x_0 be any pt in the domain.

Can we represent f as a power series?

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots + a_n(x-x_0)^n + \cdots$$

Q. What are the a_n?

$$f(x_0) = a_0 + 0 + 0 + 0 + \cdots = a_0 \quad \text{so} \quad \boxed{a_0 = f(x_0)}$$

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots \quad \boxed{a_1 = f'(x_0)}$$

$$f''(x) = 2a_2 + 3\cdot 2\cdot a_3(x-x_0) + 4\cdot 3\cdot a_4(x-x_0)^2 + \cdots$$

$$f''(x_0) = 2a_2 \quad \text{so} \quad \boxed{a_2 = \frac{1}{2!}f''(x_0)}$$

$$f'''(x) = 3\cdot 2\cdot 1\cdot a_3 + 4\cdot 3\cdot 2\, a_4(x-x_0) + \cdots \quad \boxed{a_3 = \frac{1}{3!}f'''(x_0)}$$

in general $\quad a_n = \frac{1}{n!} f^{(n)}(x_0)$!

So we the formula for the Taylor Series of f centered at $x = x_0$.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!} f^{(n)}(x_0)}_{a_n} (x-x_0)^n} \quad \circledast$$

Ex. $\boxed{f(x) = e^x}$ at $x_0 = 0$. $\qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

n	$f^{(n)}$		$f^{(n)}(0)$
0	e^x	$\rightarrow$	$e^0 = 1$
1	e^x		
2	e^x		
3	$\vdots$		
4	$\vdots$		
$\vdots$			

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$R = \infty$

Ex. $f(x) = \sin x \qquad x_0 = 0$

$$\sin x = \sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(0)}}{n!} x^n$$

n	$f^{(n)}(x)$	$\xrightarrow{x=0}$	$f^{(n)}(0)$
0	$\sin x$		0
1	$\cos x$		1
2	$-\sin x$		0
3	$-\cos x$		-1
4	$\sin x$		0
5	$\cos x$		1
$\vdots$			0
			-1

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \cdots$$

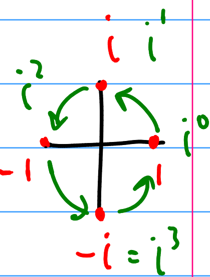
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

$R = \infty$

Ex. $f(x) = \cos x \qquad x_0 = 0$, FTIS, evens only!

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}} \leftarrow$$



<u>Ex.</u> $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ $\qquad\qquad i = \sqrt{-1}$

$$e^{it} = \sum_{n=0}^{\infty} \frac{1}{n!} (it)^n$$

$$= 1 + it + \frac{1}{2!} i^2 t^2 + \frac{1}{3!} i^3 t^3 + \frac{1}{4!} i^4 t^4 + \frac{1}{5!} i^5 t^5 + \frac{1}{6!} i^6 t^6 + \frac{1}{7!} i^7 t^7 + \cdots$$

$$= 1 + it - \frac{1}{2!} t^2 - \frac{1}{3!} i t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} i t^5 - \frac{1}{6!} t^6 - \frac{1}{7!} i t^7 + \cdots$$

$$e^{it} = \underbrace{\left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \frac{1}{6!} t^6 + - \cdots\right)}_{\cos t} + i \underbrace{\left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + - \cdots\right)}_{\sin t}$$

$$\boxed{e^{it} = \cos t + i \sin t}$$ Euler's Formula

$t = \pi$ $\qquad e^{i\pi} = \cos\pi + i \sin\pi$

$$= -1 + 0i$$

$$\boxed{e^{i\pi} + 1 = 0}$$!