M243 30 Nov 18

§11.F – Taylor Series

f smooth function (all derivatives exist) and x_0 in the domain of f.

The Taylor Series of f centered at x_0 is the power series given by

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$$

Thm. Taylor's Remainder Theorem, Lagrange Form

Let f be a function w/ Taylor Series $T(x)$ at x_0.
The N^{th} partial sum of T is a polynomial of degree at most n.

$$T_N = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}f'''(x_0)(x-x_0)^3 + \cdots + \frac{1}{N!}f^{(N)}(x_0)(x-x_0)^N$$

$$f(x) = T_N(x) + R_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{small as } N\to\infty}$$

R_N = remainder

Integral Test can be used to approximate $\sum_{n=N+1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

and the Mean Value Theorem can then be applied to give

Thm. If f has $n+1$ derivatives on an interval I containing x_0, then for x in I, there exists a number z strictly between x and x_0 satisfying

$$\boxed{R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-x_0)^{N+1}} \qquad (*)$$

Idea: Infinite sum can be replaced by a single term.

"**Problem**": We don't know z. But this formula may be used to bound the remainder.

Ex. $f(x) = e^x \qquad x_0 = 2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

n	$f^{(n)}(x)$
0	e^x
1	e^x
2	e^x
3	$\vdots$
$\vdots$	

$\xrightarrow{x=2}$ $f^{(n)}(2) = e^2$

Ex. $f(x) = 2^x \qquad x_0 = 0 \qquad$ "MacLaurin Series" when $x_0 = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!}x^n$$

n	$f^{(n)}(x)$
0	2^x
1	$\ln 2\, 2^x$
2	$(\ln 2)^2\, 2^x$
3	$(\ln 2)^3\, 2^x$
$\vdots$	$\vdots$

$f^{(n)}(x) = (\ln 2)^n 2^x \rightarrow f^{(n)}(0) = (\ln 2)^n$

$$2^x = \left(e^{\ln 2}\right)^x = \underbrace{e^{(\ln 2)x}}$$

$$\left(e^u\right)' = u' \cdot e^u$$

$$e^u = \sum_{n=0}^{\infty} \frac{1}{n!} u^n \rightarrow e^{(\ln 2)x} = \sum_{n=0}^{\infty} \frac{1}{n!} (\ln 2 \cdot x)^n$$

$$\boxed{2^x = \sum_{n=0}^{\infty} \frac{1}{n!} (\ln 2)^n x^n}$$

Ex. $f(x) = x^3$ a) $x_0 = 0$, b) $x_0 = -1$

n	$f^{(n)}(x)$
0	x^3
1	$3x^2$
2	$6x$
3	6
4	0
5	0
	$\vdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{6} f'''(x_0)(x - x_0)^3 + \frac{1}{24} f^{(4)}(x_0)(x - x_0)^4 + \cdots$$

$x_0 = 0$

$$f(x) = 0^3 + 3 \cdot 0^2 (x) + \frac{1}{2} 60 x^2 + \frac{1}{6} \cdot 6 \cdot x^3 + 0 + \cdots$$

$$f(x) = x^3$$

at $x_0 = -1$:

$$x^3 = f(x) = -1 + 3(x+1) + \frac{1}{2}(-6)(x+1)^2 + \frac{1}{6} \cdot 6 \cdot (x+1)^3$$

$$\boxed{x^3 = -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3}$$

Ex. $\sinh(x)$, $x_0 = 0$.

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

n	$f^{(n)}(x)$	$\xrightarrow{x=0}$	$f^{(n)}(0)$
0	$\sinh(x)$		0
1	$\cosh(x)$		1
2	$\sinh(x)$		0
3	$\cosh(x)$		1
4	$\vdots$		
$\vdots$			

$$\sinh(x) = x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + \cdots$$

$$\boxed{\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}}$$

OR $\quad \sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$$
\begin{aligned}
e^{x} &= \cancel{1} + x + \cancel{\tfrac{1}{2!}x^{2}} + \tfrac{1}{3!}x^{3} + \cancel{\tfrac{1}{4!}x^{4}} + \cdots \\
-\Big(e^{-x} &= \cancel{1} - x + \cancel{\tfrac{1}{2!}x^{2}} - \tfrac{1}{3!}x^{3} + \cancel{\tfrac{1}{4!}x^{4}} - + \cdots \Big)
\end{aligned}
$$

$$e^{x} - e^{-x} = 2x + 2\frac{1}{3!}x^{3} + 2\frac{1}{5!}x^{5} + \cdots$$

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right) = x + \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

Final Contender. $\quad \int e^{-x^{2}}\,dx$