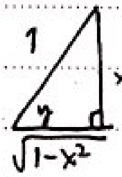
Unit II Review Guide

1. $y = \arcsin(x) \Rightarrow x = \sin(y)$

$\frac{d}{dx}\left[x = \sin(y) \right]$



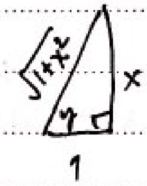
$1 = \cos(y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$

$y = \arctan(x) \Rightarrow x = \tan(y)$

$\frac{d}{dx}\left[x = \tan(y) \right]$



$1 = \sec^2(y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{(\sqrt{1+x^2})^2}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$

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2. $y = \sinh^{-1}(x)$

$\Rightarrow x = \sinh(y)$

$\Rightarrow x = \frac{1}{2}(e^y - e^{-y})$

$\Rightarrow 2x = e^y - e^{-y}$

$\Rightarrow (e^y - 2x - e^{-y} = 0)e^y$

$\Rightarrow (e^y)^2 - (2x)e^y - 1 = 0$

$\Rightarrow (e^y)^2 - (2x)e^y + x^2 = 1 + x^2$

$\Rightarrow (e^y - x)^2 = x^2 + 1$

$\Rightarrow e^y = x \pm \sqrt{x^2+1}$

$\Rightarrow e^y = x + \sqrt{x^2+1}$ since $e^y > 0$

$\Rightarrow \boxed{y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)}$

$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{d}{dx}\left[\ln\left(x + \sqrt{x^2+1}\right)\right]$

$= \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{(\sqrt{x^2+1} + x)}{(x+\sqrt{x^2+1})\sqrt{x^2+1}}$

$= \frac{1}{\sqrt{x^2+1}}$

So,

$\boxed{\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{1}{\sqrt{x^2+1}}}$

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3. a.) $\lim_{x \to \frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x} = \frac{0}{1-1} = \frac{0}{0}$ L'Hôpital

$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^+} \frac{-\sin x}{-\cos x} = \lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty.$

b.) $\lim_{t \to 0^+} \frac{\ln(t)}{t} = \frac{-\infty}{0^+} = -\infty$

3. c) $\lim_{x\to 0} \frac{\sqrt{1-3x}-\sqrt{1+8x}}{x} = \frac{\sqrt{1}-\sqrt{1}}{0} = \frac{0}{0}$ L'Hôpital

$= \lim_{x\to 0} \frac{(\sqrt{1-3x}-\sqrt{1+8x})}{x} \frac{(\sqrt{1-3x}+\sqrt{1+8x})}{(\sqrt{1-3x}+\sqrt{1+8x})} = \lim_{x\to 0} \frac{1-3x-1-8x}{x(\sqrt{1-3x}+\sqrt{1+8x})}$

$= \lim_{x\to 0} \frac{-11x}{x(\sqrt{1-3x}+\sqrt{1+8x})} \overset{L'H}{=} \lim_{x\to 0} \frac{-11}{(\sqrt{1-3x}+\sqrt{1+8x}) + x\left(\frac{-3}{2\sqrt{1-3x}} + \frac{8}{2\sqrt{1+8x}}\right)}$

$= \frac{-11}{(1+1)+0} = \frac{-11}{2}$

3. d.) $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

Apply the log: $\ln\left(\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x\right) = \lim_{x\to\infty} \ln\left(\left(1+\frac{2}{x}\right)^x\right) = \lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right)$

$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{1/x} = \frac{-\infty}{+\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{-2/x^2/(1+2/x)}{-1/x^2} = \lim_{x\to\infty} \left(1+\frac{2}{x}\right)(2) = 2$

So $\ln(y) = 2$ and $y = e^2$.

therefore $\boxed{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2}$

3.e.) $\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to\pi/2}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos x}\right)$

$= \frac{1-1}{0} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}} \left(\frac{-\cos x}{-\sin x}\right) = \lim_{x\to\pi/2} \cot(x) = 0.$

4. a.) $\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C$

	u	dv
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
	0	$\frac{1}{8}e^{2x}$

b.) $\int \arccos(x)\,dx = x\arccos(x) - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = x\arccos(x) - \frac{1}{2}\int u^{-1/2}\,du$

$u = \arccos(x)$ $\quad dv = dx$

$du = \frac{-1}{\sqrt{1-x^2}}$ $\quad v = x$

$u = 1-x^2$

$du = -2x\,dx$

$= \boxed{x\arccos(x) - \sqrt{1-x^2} + C}$

c.) $\int \ln(\sqrt{x})\,dx = \frac{1}{2}\int \ln x\,dx = \frac{1}{2}\left(x\ln x - \int 1\,dx\right)$

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad v = x$

$= \boxed{\frac{1}{2}x\ln x - \frac{1}{2}x + C}$

d.) $\int e^{\theta}\sin\theta\,d\theta = e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\,d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta\,d\theta$

$u = \sin\theta$ $\quad dv = e^{\theta}d\theta$

$du = \cos\theta\,d\theta$ $\quad v = e^{\theta}$

$u = \cos\theta$ $\quad dv = e^{\theta}d\theta$

$du = -\sin\theta\,d\theta$ $\quad v = e^{\theta}$

$\Rightarrow 2\int e^{\theta}\sin\theta\,d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta$

$\Rightarrow \boxed{\int e^{\theta}\sin\theta\,d\theta = \frac{1}{2}e^{\theta}\sin\theta - \frac{1}{2}e^{\theta}\cos\theta + C}$

e.) $\int_0^{2\pi} t\sin t\cos t\,dt$

$u = t$ $\quad dv = \sin t\cos t\,dt$

$du = dt$ $\quad v = \frac{1}{2}\sin^2 t\,dt$

$= \frac{1}{2}t\sin^2 t - \frac{1}{2}\int \sin^2 t\,dt = \frac{1}{2}t\sin^2 t - \frac{1}{4}\int 1 - \cos(2t)\,dt$

$= \frac{1}{2}t\sin^2 t - \frac{1}{4}t + \frac{1}{8}\sin(2t)\Big|_0^{2\pi} = -\frac{1}{4}(2\pi) = \boxed{\frac{-\pi}{2}}$

f.) $\int \frac{z}{\sqrt{z^2-9}}\,dz = \frac{1}{2}\int u^{-1/2}\,du = \sqrt{u} + C = \boxed{\sqrt{z^2-9} + C}$

$u = z^2-9$

$du = 2z\,dz$

g.) $\int \frac{17}{(x-1)(x+1)^2}\,dx = 17\left(\int \frac{A}{x-1}dx + \int \frac{B}{x+1}dx + \int \frac{C}{(x+1)^2}dx\right) = 17A\ln|x-1| + 17B\ln|x+1| - \frac{17C}{x+1} + \text{Constant.}$

PFD:

$1 = A(x+1)^2 + B(x^2-1) + C(x-1)$

$1 = Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$

$\begin{cases} 0 = A+B \Rightarrow B = -A \\ 0 = 2A + C \\ 1 = A - B - C \Rightarrow 1 = 2A - C \Rightarrow 1 = -2C \end{cases}$

$\left.\begin{array}{l} C = -\frac{1}{2} \\ A = +\frac{1}{4} \\ B = -\frac{1}{4} \end{array}\right\}$

Sol'n:

$\boxed{\int \frac{17}{(x-1)(x+1)^2}\,dx = \frac{17}{4}\ln|x-1| - \frac{17}{4}\ln|x+1| + \frac{17}{2(x+1)} + C}$

h.) $\int_0^{\pi} e^{\cos t}\sin(2t)\,dt = 2\int_0^{\pi} e^{\cos t}\cos t\,\sin t\,dt = -2\int_1^{-1} e^{\theta}\cdot\theta\,d\theta = 2\int_{-1}^{1} e^{\theta}\theta\,d\theta$

$\theta = \cos t$ $\quad\theta(\pi) = -1$ $\quad u = \theta$ $\quad dv = e^{\theta}d\theta$

$d\theta = -\sin t\,dt$ $\quad\theta(0) = 1$ $\quad du = d\theta$ $\quad v = e^{\theta}$

$= 2\int_{-1}^{1} e^{\theta}\theta\,d\theta = 2\left(\theta e^{\theta} - \int e^{\theta}d\theta\right)\Big|_{-1}^{1} = 2\left(\theta e^{\theta} - e^{\theta}\right)\Big|_{-1}^{1} = 2\left(1e - e + e^{-1} + e^{-1}\right) = \boxed{\frac{4}{e}}$

i.) $\int \tan^3\theta\sec^6\theta\,d\theta = \int u^3(u^2+1)^2\,du = \int u^3(u^4+2u^2+1)\,du = \int u^7 + 2u^5 + u^3\,du$

$u = \tan\theta$ $\quad = \frac{1}{8}u^8 + \frac{2u^6}{6} + \frac{1u^4}{4} + C$

$du = \sec^2\theta\,d\theta$

$\sec^4\theta = (\sec^2\theta)^2 = (\tan^2\theta + 1)^2 = (u^2+1)^2$ $\quad = \boxed{\frac{1}{8}\tan^8\theta + \frac{1}{3}\tan^6\theta + \frac{1}{4}\tan^4\theta + C}$

j.) $\int \sin^2 x\cos^2 x\,dx = \int \frac{1}{8} - \frac{1}{8}\cos(4x)\,dx$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\quad = \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}$

$\sin^2 x\cos^2 x = \frac{1}{4}\left(1 - \cos^2(2x)\right)$

$= \frac{1}{4}\left(1 - \frac{1}{2}(1 + \cos(4x))\right)$

$= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4x)$

$= \frac{1}{8} - \frac{1}{8}\cos(4x)$

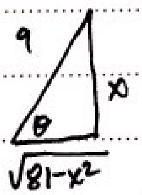
k.) $\int \frac{x^2}{\sqrt{81 - x^2}}\,dx = \int \frac{81\sin^2\theta\cdot\cancel{9\cos\theta}}{\cancel{9}\,\cancel{\cos\theta}}\,d\theta = \frac{81}{2}\int 1 - \cos 2\theta\,d\theta$

$81 - x^2 = 81\left(1 - \left(\frac{x}{9}\right)\right)^2$ $\quad = \frac{81}{2}\theta - \frac{81}{4}\sin(2\theta) + C$

Put $\frac{x}{9} = \sin\theta$ $\quad = \frac{81}{2}\theta - \frac{81}{2}\sin\theta\cos\theta + C$

$x = 9\sin\theta$

$dx = 9\cos\theta\,d\theta$



$= \frac{81}{2}\arcsin\left(\frac{x}{9}\right) - \frac{\cancel{81}}{2}\cdot\frac{x}{\cancel{9}}\cdot\frac{\sqrt{81-x^2}}{\cancel{9}} + C$

$= \boxed{\frac{81}{2}\arcsin\left(\frac{x}{9}\right) - \frac{1}{2}x\sqrt{81 - x^2} + C}$

$\sin\theta = \frac{x}{9}$

$\cos\theta = \frac{\sqrt{81-x^2}}{9}$

l.) $\int_0^7 \sqrt{x^2+49}\, dx \quad = \int_0^{\pi/4} 7\sec\theta \cdot 7\sec^2\theta\, d\theta = 49\int_0^{\pi/4} \sec\theta \sec^2\theta\, d\theta$

$x^2+49 = 49\left(\left(\frac{x}{7}\right)^2+1\right)$

$u = \sec\theta \qquad dv = \sec^2\theta\, d\theta$

Put $\frac{x}{7} = \tan\theta$

$du = \sec\theta\tan\theta\, d\theta \qquad v = \tan\theta$

$x = 7\tan\theta \quad \rightarrow \quad \theta = \arctan\left(\frac{x}{7}\right)$

$dx = 7\sec^2\theta\, d\theta \qquad \theta(7) = \arctan(1) = \pi/4, \ \theta(0) = \arctan(0) = 0.$

$= 49\left(\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta\right) = 49\left(\sec\theta\tan\theta - \int\sec\theta(\sec^2\theta - 1)\, d\theta\right)$

$= 49\left(\sec\theta\tan\theta + \int\sec\theta\, d\theta - \int\sec^3\theta\, d\theta\right)$

$\Rightarrow 49\int\sec^3\theta\, d\theta = \frac{49}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4}$

$= \frac{49}{2}\left(\sqrt{2}\cdot 1 + \ln|\sqrt{2}+1| - 0 - \ln 1\right) = \boxed{\frac{49}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)}$

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m.) $\int \frac{7t-5}{t+5}\, dt \quad = \int 7 - \frac{40}{t+5}\, dt = \boxed{7t - 40\ln|t+5| + C}$

synth. div.:

$$\begin{array}{r|rr} -5 & 7 & -5 \\ & \downarrow & -35 \\ \hline & 7 & \boxed{-40} \end{array}$$

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n.) $\int \frac{25}{v^3-8}\, dv = \int \frac{25}{(v-2)(v^2+2v+4)}\, dv = \underbrace{\frac{1}{12}\int\frac{25}{v-2}\, dv}_{} - \frac{1}{12}\int\frac{25v-100}{v^2+2v+4}\, dv$

$u = v^2+2v+4$

$du = 2v+2$

$= \frac{25}{12}\ln|v-2|$

$25 = A(v^2+2v+4) + (Bv+C)(v-2)$

cts:

$(v^2+2v+1)+4-1$

$v=2: \quad 25 = 12A$

$(v+1)^2+3$

$u = \frac{v+1}{\sqrt{3}}$

$A = \frac{25}{12}$

$3\left(\left(\frac{v+1}{\sqrt{3}}\right)^2 + 1^2\right)$

$du = \frac{1}{\sqrt{3}}dv$

$v^2: \ A + B = 0$

$v: \ 2A - 2B + C = 0$

$\#: \ 4A - 2C = 25$

$B = \frac{-25}{12}$

$2C = 25 - \cancel{4}\left(\frac{25}{\cancel{12}_3}\right)$

$2C = \frac{50}{3}$

$C = \frac{25}{3} = \frac{100}{12}$

$= \frac{25}{12}\ln|v-2| - \frac{25}{12}\int\frac{v-4}{v^2+2v+4}\, dv$

$= \frac{25}{12}\ln|v-2| - \frac{25}{12}\frac{1}{2}\int\frac{2v+2}{v^2+2v+4}\, dv - \frac{25}{12}\frac{1}{2}\int\frac{-6}{v^2+2v+4}\, dv$

$= \frac{25}{12}\ln|v-2| - \frac{25}{24}\ln|v^2+2v+4| + \frac{25}{12}\frac{1}{\cancel{2}}\int\frac{+6}{3\left(\left(\frac{v+1}{\sqrt{3}}\right)^2+1\right)}\, dv$

$= \boxed{\frac{25}{12}\ln|v-2| - \frac{25}{24}\ln|v^2+2v+4| + \frac{25\sqrt{3}}{12}\arctan\left(\frac{v+1}{\sqrt{3}}\right) + C}$

5. Should be: $\int_1^4 \frac{1}{x^2}\,dx$

$a=1$

$b=4$

$n=6$

$dx = \frac{4-1}{6} = \frac{1}{2}$

$f(x) = 1/x^2$

k	x_k	$f(x_k)$	Simp
0	1	1	1
1	$3/2$	$1/(9/4) = 4/9$	$16/9$
2	2	$1/4$	$2/4$
3	$5/2$	$1/(25/4) = 4/25$	$16/25$
4	3	$1/9$	$2/9$
5	$7/2$	$1/(49/4) = 4/49$	$16/49$
6	4	$1/16$	$1/16$

$\Sigma = \frac{88769}{19600}$

$S_6 = \frac{1}{2}\cdot\frac{1}{3}\cdot\Sigma = \frac{88769}{117600} \approx \boxed{0.754838}$

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6. a.) $\int_8^9 \frac{x}{x-9}\,dx = \lim_{t\to 9^-} \int_8^t \frac{x}{x-9}\,dx = \lim_{t\to 0^-} \int_{-1}^t \frac{u+9}{u}\,du = \lim_{t\to 0^-} \int_{-1}^t 1 + \frac{9}{u}\,du$

$u = x-9 \to x = u+9$

$du = dx$

$= \lim_{t\to 0^-} t + 9\ln|t| - (-1 + 9\ln 1)$

$= 0 + 9\underbrace{\lim_{t\to 0^-} \ln|t|}_{-\infty} + 1 + 0$

Diverges!

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b.) $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_0^{\ln t} u\,du = \lim_{t\to\infty} \frac{1}{2}u^2 \Big|_0^{\ln t}$

$u = \ln x$ $\quad u(t) = \ln t$

$du = \frac{1}{x}dx$ $\quad u(1) = 0$

$= \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \frac{1}{2}\infty^2 = \infty$

Diverges.

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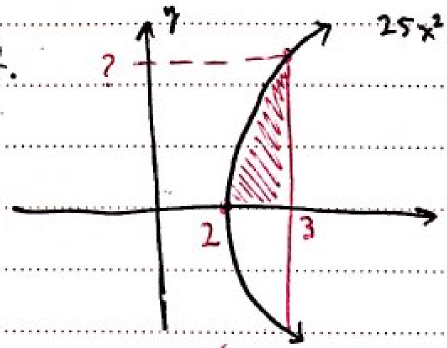
c.) $\int_1^\infty \frac{1}{t^2}\,dt = \lim_{u\to\infty} \int_1^u \frac{1}{t^2}\,dt = \lim_{u\to\infty} \left(\frac{-1}{t}\right)\Big|_1^u = \lim_{u\to\infty}\left(\frac{-1}{u} + 1\right) = 0 + 1 = 1.$

converges!

d.) $\int_0^1 \frac{1}{t^2}\,dt = \lim_{u\to 0^+} \int_u^1 \frac{1}{t^2}\,dt = \lim_{u\to 0^+} \left(\frac{-1}{u}+1\right) = -\infty + 1 = -\infty$

Diverges!

7.



$25x^2 - 4y^2 = 100 \rightarrow x^2 = 4 + \frac{4y^2}{25} \rightarrow x^2 = 4\left(1+\left(\frac{y}{5}\right)^2\right)$

$\text{Area} = 2\int_0^? 3 - \sqrt{1+\left(\frac{y}{5}\right)^2}\,dy$

$?: \frac{9}{4} = 1 + \left(\frac{y}{5}\right)^2 \Rightarrow \frac{5}{4} = \frac{y^2}{25} \Rightarrow y = \sqrt{\frac{125}{4}} = \frac{5\sqrt{5}}{2}$

$2\int_0^{\frac{5\sqrt5}{2}} 3 - \sqrt{1+(y/5)^2}\,dy = 2\int_0^{\frac{5\sqrt5}{2}} 3\,dy - 2\int_0^{\frac{5\sqrt5}{2}} \sqrt{1+(y/5)^2}\,dy$

$\frac{y}{5} = \tan\theta \qquad \theta = \arctan(y/5)$

$y = 5\tan\theta \qquad \theta\left(\frac{5\sqrt5}{2}\right) = \arctan\left(\frac{\sqrt5}{2}\right)$

$dy = 5\sec^2\theta\,d\theta \qquad \theta(0) = \arctan(0) = 0$

$= 2\cdot 3\cdot \frac{5\sqrt5}{2} - 2\int_0^{\arctan\left(\frac{\sqrt5}{2}\right)}$

$25x^2 - 4y^2 = 100 \rightarrow 4y^2 = 25x^2 - 100$

$y^2 = \frac{25}{4}x^2 - 25$

$y^2 = 25\left(\left(\frac{x}{2}\right)^2 - 1\right)$

Retry! $\text{Area} = 2\int_?^3 \sqrt{25\left(\left(\frac{x}{2}\right)^2-1\right)}\,dx = 10\int_2^3 \sqrt{\left(\frac{x}{2}\right)^2-1}\,dx$

x, $\sqrt{x^2-4}$, θ, 2

$\frac{x}{2} = \sec\theta \qquad \theta = \arccos\left(\frac{2}{x}\right)$

$x = 2\sec\theta \qquad \theta(3) = \arccos\left(\frac{2}{3}\right)$

$dx = 2\sec\theta\tan\theta\,d\theta \qquad \theta(2) = \arccos(1) = 0$

$\arccos\left(\frac{2}{3}\right) = \alpha$

$= 10\int_0^{\alpha} \tan\theta \cdot 2\sec\theta\tan\theta\,d\theta = 20\int_0^{\alpha} \sec^3\theta\,d\theta - 20\int_0^{\alpha} \sec\theta\,d\theta$

$= \text{by previous work} \rightarrow 20\left(\frac{\sec\theta\tan\theta}{2} + \frac{\ln|\sec\theta+\tan\theta|}{2} - \ln|\sec\theta+\tan\theta|\right)$

$= 10\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\alpha} = 10\left(\frac{x}{2}\frac{\sqrt{x^2-4}}{2} - \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|\right)\Big|_2^3$

$= 10\left(\frac{3\sqrt5}{4} - \ln\left(\frac{3+\sqrt5}{2}\right)\right) - 10\left(1\cdot 0 - \ln|1+0|\right) = \boxed{\frac{15\sqrt5}{2} - 10\ln\left(\frac{3+\sqrt5}{2}\right)}$

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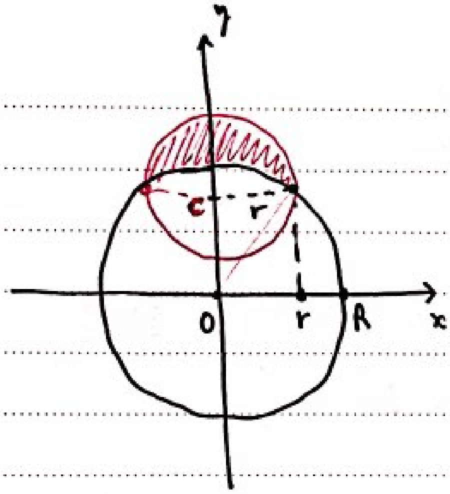
8.



top: $x^2+(y-c)^2=r^2$

$y = c+\sqrt{r^2-x^2}$

bott: $x^2+y^2=R^2$

$y=\sqrt{R^2-x^2}$

$c = y_{bott}(r) = \sqrt{R^2-r^2}$

So $y_{top} = \sqrt{R^2-r^2}+\sqrt{r^2-x^2}$, $y_{bott} = \sqrt{R^2-x^2}$

$$\text{Area} = 2\int_0^r \left(\sqrt{R^2-r^2}+\sqrt{r^2-x^2}\right)-\sqrt{R^2-x^2}\,dx$$

$$= 2\int_0^r \sqrt{R^2-r^2}\,dx + 2\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r\sqrt{R^2-x^2}\,dx$$

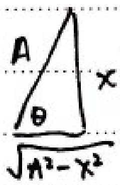
Compute once: $\int_0^r \sqrt{A^2-x^2}\,dx$, $A \geq r$.

$A^2-x^2 = A^2\left(1-\left(\frac{x}{A}\right)^2\right)$

Put $\frac{x}{A} = \sin\theta$

$x = A\sin\theta$

$dx = A\cos\theta\,d\theta$



$$= \int A\cos\theta \cdot A\cos\theta\,d\theta = \frac{A^2}{2}\int 1+\cos(2\theta)\,d\theta$$

$$= \frac{A^2}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Big|_0^r = \frac{A^2}{2}\theta + \frac{A^2}{2}\sin\theta\cos\theta\Big|$$

$$= \frac{A^2}{2}\left(\arcsin\left(\frac{x}{A}\right)+\frac{x}{A}\frac{\sqrt{A^2-x^2}}{A}\right)\Big|_0^r$$

$$= \frac{A^2}{2}\left(\arcsin\left(\frac{r}{A}\right)-0+\frac{r\sqrt{A^2-r^2}}{A^2}-0\right)$$

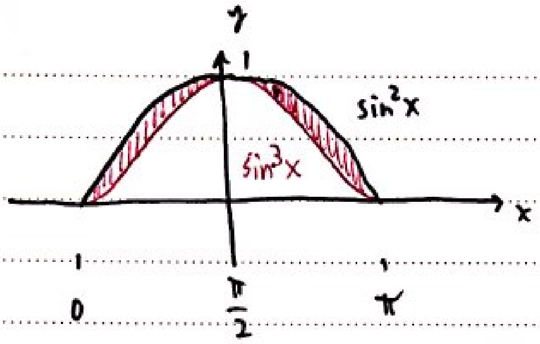
$$= \frac{A^2}{2}\arcsin\left(\frac{r}{A}\right)+\frac{1}{2}r\sqrt{A^2-r^2}.$$

Plugging in,

$$\text{Area} = \cancel{2}\left(2r\sqrt{R^2-r^2}+\frac{r^2}{\cancel{2}}\underbrace{\arcsin(1)}_{\frac{\pi}{2}}+\underbrace{\cancel{\frac{1}{2}r\sqrt{r^2-r^2}}}_{0}-\left(\frac{R^2}{\cancel{2}}\arcsin\left(\frac{r}{R}\right)+\frac{1}{\cancel{2}}r\sqrt{R^2-r^2}\right)\right)$$

$$= \boxed{r\sqrt{R^2-r^2}+\frac{\pi r^2}{2}-R^2\arcsin\left(\frac{r}{R}\right)}$$

9.



$$\text{Area} = 2\int_0^{\pi/2} \sin^2 x - \sin^3 x \, dx$$

$$= 2\int_0^{\pi/2} \sin^2 x \, dx - \int_0^{\pi/2} \sin^2 x \sin x \, dx$$

$$= \frac{2}{2}\int_0^{\pi/2} 1 - \cos 2x \, dx - \int_0^1 (1-u^2) \, du$$

$u = \cos x$ $\quad u(\pi/2) = 0$

$du = -\sin x dx$ $\quad u(0) = 1$

$$= \left(x - \tfrac{1}{2}\sin(2x)\right)\Big|_0^{\pi/2} - \left(u - \tfrac{1}{3}u^3\right)\Big|_0^1$$

$$= \pi/2 - 0 - \tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 0 - \left(1 - 0 - \tfrac{1}{3} + 0\right)$$

$$= \frac{\pi}{2} - \frac{2}{3} = \boxed{\frac{3\pi - 4}{6}}$$

10. $|E_{S_n}| \le \dfrac{K(b-a)^5}{180 n^4}$, $K \le \max\limits_{a \le x \le b} \left(f^{(4)}(x)\right)$.

$f(x) = e^{-2x^2}$

$f'(x) = -4x e^{-2x^2}$

$f''(x) = -4 e^{-2x^2} + 16x^2 e^{-2x^2}$

$f'''(x) = 16x e^{-2x^2} + 32x e^{-2x^2} - 64x^3 e^{-2x^2} = 48 x e^{-2x^2} - 64x^3 e^{-2x^2}$

$f^{(4)}(x) = 48 e^{-2x^2} - 192x e^{-2x^2}$

$-192 x^2 e^{-2x^2} + 256x^4 e^{-2x^2}$

$= (256x^4 - 384x^2 + 48) e^{-2x^2}$

Max happens when $x = 0$: $f^{(4)}(0) = 48$.

So, $\dfrac{48(2-0)^5}{180 n^4} \le \dfrac{1}{1000} \Rightarrow \dfrac{180 n^4}{1536} \ge 1000 \Rightarrow n^4 \ge \dfrac{1536000}{180}$

$\Rightarrow n \ge \sqrt[4]{\dfrac{1536000}{180}} \approx 9.61125$ so choose $\boxed{n = 10}$